Name: ____________________

Physics 7502 Quantum Mechanics Spring 2019

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Quiz 2

Given: Friday, Jan 25, 2019   Room: Scott N050   Time: 11.30-12.25 am

Problem 1: (10 points) Consider a particle of mass m in an infinite square well with potential

$$\begin{aligned} V(x) &= 0, \quad -a < x < a \\ &= \infty, \quad |x| > a \end{aligned} \tag{1}$$

(i) Find the ground state energy eigenfunction $\psi_0(x)$, and normalize it. (5 points)

(ii) Now we add a small perturbation to the potential so that $V(x) \to V(x) + \epsilon V_1(x)$, where

$$\begin{aligned} V_1(x) &= V_0, \quad -b < x < b \\ &= 0, \quad |x| > b \end{aligned} \tag{2}$$

with $0 < b < a$. Find the shift in the energy of the ground state under this perturbation, to first order in ϵ. (5 points).

Solution: : (i) We have

$$\psi_0 = C \cos kx \tag{3}$$

$$\frac{\hbar^2}{2m} k^2 = E_0 \tag{4}$$

$$\cos ka = 0, \quad k = \frac{\pi}{2a} \tag{5}$$

Thus

$$\psi_0 = C \cos(\frac{\pi}{2a} x) \tag{6}$$

$$|C|^2 \int_{x=-a}^{a} dx \cos^2 kx = |C|^2 \int_{x=-a}^{a} dx \frac{1}{2}(1 + \cos(2kx)) = |C|^2 \frac{1}{2}(x + \frac{1}{2k}\sin(2kx))|_{-a}^{a} = a|C|^2 \tag{7}$$

Setting this to unity gives

$$C = \frac{1}{\sqrt{a}} \tag{8}$$

and

$$\psi_0(x) = \frac{1}{\sqrt{a}} \cos(\frac{\pi}{2a} x) \tag{9}$$

(ii) We get

$$E^{(1)} = \int_{x=-b}^{b} dx V_0 \frac{1}{a} \cos^2 kx = \frac{V_0}{a} \int_{x=-b}^{b} dx \frac{1}{2}(1 + \cos(2kx)) \tag{10}$$

$$= \frac{V_0}{2a}(x + \frac{1}{2k}\sin(2kx))|_{-b}^{b} = \frac{V_0}{2a}(2b + \frac{1}{2k}(\sin(2kb) - \sin(-2kb))) \tag{11}$$

$$= \frac{V_0}{2a}(2b + \frac{1}{2k}(2\sin(2kb))) = \frac{V_0}{a}(b + \frac{a}{\pi}\sin(\frac{\pi b}{a})) \tag{12}$$

Thus the shift is

$$\Delta E = \epsilon E^{(1)} = \epsilon V_0(\frac{b}{a} + \frac{1}{\pi}\sin(\frac{\pi b}{a})) \tag{13}$$

Problem 2: (10 points) (a) A Hydrogen atom is kept in an electric field $\mathcal{E}$ pointing along the $\hat{z}$ direction. Find the perturbation Hamiltonian $\hat{H}^{(1)}$. (5 points)

(b) You are given the following matrix elements for the wavefunctions $|n, l, m\rangle$:

$$\langle 2,0,0|r\cos\theta|2,0,0\rangle = 0 \tag{14}$$

$$\langle 2,0,0|r\cos\theta|2,1,0\rangle = -3\frac{\hbar^2(4\pi\epsilon_0)}{me^2} \tag{15}$$

$$\langle 2,1,0|r\cos\theta|2,1,0\rangle = 0 \tag{16}$$

Find the separation ΔE between the two energy levels that are split by the Stark effect in an electric field $\mathcal{E}$, in terms of the variables given above in the problem. (5 points)

Solution: : (a) We have

$$V = -e\Phi = e\mathcal{E}z = e\mathcal{E}r\cos\theta \tag{17}$$

(b) We have

$$\begin{pmatrix} \langle 2,0,0|\hat{H}^{(1)}|2,0,0\rangle & \langle 2,0,0|\hat{H}^{(1)}|2,1,0\rangle \\ \langle 2,1,0|\hat{H}^{(1)}|2,0,0\rangle & \langle 2,1,0|\hat{H}^{(1)}|2,1,0\rangle \end{pmatrix} = -3\frac{\hbar^2(4\pi\epsilon_0)}{me^2}e\mathcal{E}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18}$$

The eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{19}$$

are

$$\lambda = 1 \ : \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda = -1 \ : \ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{20}$$

Thus the energies are

$$E = E_2^{(0)} - 3\frac{\hbar^2(4\pi\epsilon_0)}{me^2}e\mathcal{E}, \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|2,0,0\rangle + |2,1,0\rangle) \tag{21}$$

$$E = E_2^{(0)} + 3\frac{\hbar^2(4\pi\epsilon_0)}{me^2}e\mathcal{E}, \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|2,0,0\rangle - |2,1,0\rangle) \tag{22}$$

Thus the splitting is

$$\Delta E = 6\frac{\hbar^2(4\pi\epsilon_0)}{me^2}e\mathcal{E} \tag{23}$$