Graduate student understanding of quantum mechanical spin

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A framework of cyclic observation and triangulation was applied over a period of 4 years to graduate student difficulties related to quantum spin, in which numerous in-class observations and interviews were used to identify common, persistent difficulties. Written items were iteratively developed over two years to add a quantitative component. Items were administered to graduate students at two collaborating institutions, over three years. We find that students generally obtained scores or correct proportions ranging from 30%-70% on the written items, and answering patterns were similar across all institutions. All items were identified by the course instructors as being relevant to instructional goals of the course. We report on a number of graduate student difficulties with spin, including orthogonality of spin-1/2 states, projections of spin states, spin addition, and exchange symmetry. We briefly discuss possible theoretical frameworks through which to interpret these results.

I. INTRODUCTION

In this paper we investigate graduate student understanding of several aspects of spin in graduate level quantum mechanics (QM). Quantum mechanical spin is ubiquitous in graduate QM courses and textbooks [1,2] and has critical applications in particle, atomic, nuclear, and condensed matter physics and in emerging areas of physics such as spintronics (see for example [3, 4]). Further, the phenomenon and concept of quantum mechanical spin may also have great pedagogical utility. For example, spin-1/2 particles are often used to learn about two-level systems; they offer a low-dimensional vector space in which to learn the linear algebra of quantum mechanics [1]. Some undergraduate QM textbooks in fact begin with simple spin ½ systems instead of wave functions [5], and some researchers, such as Sadaghiani [6,7], argue that such "spins first" curricula allow students to develop facility with quantum mechanics before confronting difficult calculus, or the way such systems map onto our classical understanding.

Although physics education researchers have been studying student difficulties in QM for decades (for a review see [8]; for early work see [9, 10]), and have been developing improved instructional materials [10-15], relatively few have studied student understanding of spin, specifically. One notable exception is Brown and Singh [16], in which undergraduate student understanding of basic time-dependence was studied in the context of Larmor precession. Most existing work has been done with undergraduate populations, with only a few publications dealing with graduate-level QM [15, 17-22], and until recently only one of those investigated graduate student understanding of spin [22]. In that work, Zhu and Singh primarily focused on the effects of a Quantum Interactive Learning Tutorial (QuILT) designed to improve student understanding of the Stern-Gerlach experiment and consequences for spin.

A number of conceptual assessments exist that address spin to varying extents [17,23]. More recently, a paper by Marshman and Singh [24] introduced the Quantum Mechanics Formalism and Postulates Survey (QMFPS), which has broader coverage of spin (eleven questions), including measurement, expectation values, simultaneous eigenstates, the Stern Gerlach experiment, and time-dependence, among others. Although the QMFPS was given to graduate students as well as undergraduates; it was not given as a pretest or posttest, but rather administered halfway into a graduate quantum mechanics class.

To the extent that some of the topics investigated in this study overlap with the above studies mentioned, we will note the extent to which this study confirms those findings. However, our study led us to investigate several topics in quantum spin in which there are no previous studies, yet which were seen as relevant and important by the instructors in the study. This includes student understanding of basic concepts in orthogonality, projections, and exchange symmetry in the context of spin. Therefore, the goal of this work is to add to the emerging picture of graduate student understanding of these critical concepts and properties of quantum spin. Further, in this study we are interested not only in the status quo of graduate student understanding, but also of the effect of "typical" graduate instruction on this student understanding. We will use as a proxy for "typical instruction" the results observed from several instructors at two institutions over several semesters.

The research questions for this study are (1) what are the important yet currently unidentified graduate student difficulties with understanding essential concepts related to quantum spin? (2): To what extent does student understanding of these topics change over the course of typical graduate instruction, as observed with several instructors and institutions? Here, we will present some of the most common student errors and comment on changes in performance from pretests to posttests.

II. METHODS

In identifying student difficulties, we employed an iterative process of observing students, constructing and field-testing questions with multiple response modalities, consulting with instructors to ensure validity and course relevance, and finally refining the question for the next iteration. This general method of systematically identifying and describing student difficulties is one that has been used extensively in physics education research, including in the field of student understanding of concepts in quantum mechanics [8, 17, 25, 26].

Initial observation of students took place in lectures and in guided group work sessions which were research-based voluntary attendance sessions designed to help student with difficult material in the course [15]. The group work session field notes were much richer than the lecture notes since the group work context naturally facilitates discussion, but also because group work activities were designed to confront potential misunderstandings. Field notes described common student errors, as well as student justifications of the errors and discussions that led to their resolution.

Once specific student difficulties had emerged, initial versions of items were written to investigate the misunderstandings in more depth. Various combinations of these items were administered as both pretests and posttests for the first and second semester to graduate students enrolled in graduate QM at The Ohio State University and at the University of Cincinnati.

Student responses to items were coded as fitting into one or more categories of student difficulty. This was initially done by two researchers with an inter-rater reliability over all items of 80%, which rose to 95% upon discussion and refinement of coding criteria.

In the second year that pre and posttest items were used, 25 student interviews were also conducted These interviews

took about 45 minutes. Students were asked to explain their reasoning after writing responses to questions. This is related to the Think-Aloud protocol [27], but differs in requiring students to first solve a problem on paper.

This work focuses on four of the assessment items that are related to quantum mechanical spin. These items are listed below, with correct answers written out or bolded.

1. Suppose you are working on spin-1/2 particles, and you measure a certain spin to be in the state $|+\rangle_y$. Which of the following states is/are orthogonal to this? Circle all that apply.
   (a) $|-\rangle_y$ (b) $|+\rangle_x$ (c) $|-\rangle_x$ (d) $|+\rangle_z$ (e) $|-\rangle_z$

2. Suppose a spin-1/2 particle is sent through a Stern Gerlach device (inhomogeneous magnetic field) and a measurement is made of the particle's deflection that shows the state must be the eigenstate of $\hat{S}_x$: $|\psi\rangle = |+\rangle_x$, giving it an eigenvalue of $+\hbar/2$. Immediately afterwards, what is the _magnitude_ of the projection of the state $|\psi\rangle$ onto each of the following states?

(a) $|+\rangle_z$ _____ (answer: $1/\sqrt{2}$)
(b) $|-\rangle_z$ _____ (answer: $1/\sqrt{2}$)
(c) $|-\rangle_x$ _____ (answer: 0)
(d) $|+\rangle_y$ _____ (answer: $1/\sqrt{2}$)
(e) $|-\rangle_y$ _____ (answer: $1/\sqrt{2}$)

3. Consider two spin-1/2 fermions (A and B). At a particular time, in the uncoupled (z) basis, their spin states are $|m_A = 1/2\rangle$ and $|m_B = -1/2\rangle$. If a measurement is made (immediately, with negligible passage of time) of the total spin of the system s_{Tot}, what possible values might be obtained? (answer: 0 or 1, in units of $\hbar$)

4. Two spin-1 bosons are together in a harmonic oscillator potential. Your colleague has prepared them in a particular state, but he/she has not told you what that state is. Below are several statements about what the states could possibly be. Circle all statements that are FALSE.

(a) The total spin angular momentum of the system could be $s_{total} =1$, but this would require the two bosons to be in different spatial energy states.

(b) If they are both in the spatial HO ground state, their spin state can either be symmetric or antisymmetric under exchange, since there is no Pauli Exclusion Principle for bosons.

(c) Both particles must be in the ground state and have the same spin state because they are bosons.

(d) The bosons cannot occupy the first excited state, because that state is spatially antisymmetric.

(e) One boson could be in the ground state, and the other could be in the second excited state, but this requires that their spin states be antisymmetric under exchange.

(f) Bosons must always be in a spin state that is symmetric under exchange.

These items and distractors were directly based on observations of student difficulties in group discussions. For example, we observed in class that graduate students would struggle with inner products of spin states even when the

answers are simple (as when the states are orthogonal). This was in stark contrast to student work with inner products between different energy eigenstates in 1-D wells, with which they had no difficulty. There were indicators of students confusing orthogonality in the Hilbert space with orthogonality in the Cartesian space, but these indicators were not consistent over time or between students. Multiple students claimed, " $|+\rangle_x$ is orthogonal to $|+\rangle_z$". One student posed the question, "If $|+\rangle_x$ has a non-zero projection onto $|+\rangle_z$, then in what sense is $|+\rangle_x$ along $+x$?" Items 1 and 2 were developed to study this further.

Item 4 was similarly motivated. In early interviews with graduate students, a number of misunderstandings presented themselves on the topic of spin in the context of exchange symmetry. For example, when asked about the symmetry of the spin part of a bosonic state, several students indicated that it must "always be symmetric". When asked about the spatial part of the wavefunction, students often either answered that it didn't matter, or that it too must be symmetric under exchange. This indicated that the reliance of exchange symmetry on the full wavefunction (spatial and spin parts) is not clear to some students. There were more extreme assertions; for example, one student simply replied "No, all bosons must be in the ground state. That's what it means to be a boson." This may be attributable to the fact that most students know of bosons in the context of Bose-Einstein condensates, in which all particles occupy the same state. By far the most common error in interviews was the conflation of particle exchange symmetry with reflection symmetry (parity) of the spatial portion of the wavefunction. For example, students would say that bosons cannot occupy the first excited state in a harmonic oscillator potential, "because it is antisymmetric", not referring to the relative sign between exchange terms in the two-particle state, but referring to the reflection symmetry of that spatial wave function in a single-particle oscillator. Item 4 was drafted to gauge the prevalence of these misconceptions. The resulting question is relatively difficult, in the sense that a number of potentially distracting factors must be well-understood in order for students to answer the problem entirely correctly.Core course instructors were shown the items prior to their use. Items were only used if instructors approved of them, and agreed the items were in line with instructional goals of the course. The number of cohorts and students given each item in this study are organized in Table 1.

Participating students were awarded a small amount of course credit, either as an in-class assessment of learning, or as a flexibly-scheduled homework assignment, with full credit for participation. Students were given the opportunity to opt out of participation in research with no penalty. A total of 93% of enrolled students completed the task and agreed to participate in research, and the remaining students have had their data removed from all analysis in this work.

III. RESULTS AND DISCUSSION

The overall scores or proportion of correct answers for each item by institution/cohort are presented in Table 1. Items 1-3 were scored as correct or incorrect. The ϕ statistic is shown for each item, indicating the significant correlations between pretest and posttest correctness for items 1 and 2. Results also indicate that pretest and posttest performances were not statistically different, and the posttest performance was not different between cohorts for items 1-3., indicating that cohort data may be combined for analysis. Item 4 was more complex and was scored according to whether each answer option was correctly selected/unselected. A two-way repeated measures ANOVA indicates that significant main effects of differences between cohort and time, and a cohort-time interaction effect. Examination of that data reveals the interaction is due to one cohort having a large gain, while the other two cohorts have essential zero gains.

TABLE 1. Summary and comparisons of pre and posttest performance for all items. Here "AU##" and "SP##" refer to autumn and spring semesters, respectively, of the year 20##. Items 1-3 were scored binary, such that the quantity of interest is the percentage of answers that were correct. Pre/post comparisons are made using a McNemar test; comparisons between cohorts were made using a Fisher exact test. Item 4 was scored closer to a continuum; pre and posttest mean scores are shown. A two-way ANOVA was applied to these, with the main effects being time and cohort. Statistics with $p < 0.05$ appear in bold.

Item	Institution	Cohort	N	Correct Pre (%)	Correct Post (%)	Pre/post ϕ (all)	Pre/post McNemar p (all)	Cohort Fisher Ex. p on post
	1	AU16	26	65	65			
1	1	AU17	35	57	71	**0.53**	0.33	0.87
	2	AU17	17	65	65			
2	1	AU17	31	29	42	**0.60**	1.0	-
	1	SP17	30	30	53			
3	1	SP18	30	43	50	0.11	0.11	0.65
	2	SP18	10	60	70			

Item	Institution	Cohort	N	Score Pre (%)	Score Post (%)	95% CI gains (all)	Time p (all)	Cohort $\times$ time p
	1	SP17	28	58	60			
4	1	SP18	29	61	59	**[1.7, 16]**	**0.016**	**0.014**
	2	SP18	13	47	73			

Regarding the correct answer to Item 1, the only spin state that is orthogonal to $|+\rangle_y$ is $|-\rangle_y$. Most students correctly indicated this on the pre and post tests as shown in Table 1. But quite a few students claimed that eigenstates of S_x and S_z were all orthogonal to $|+\rangle_y$, consistent with confusion between orthogonality in the Hilbert space of spin states, and orthogonality of Cartesian axes used to label eigenstates (22% pre, 10% post).

In item 2, several incorrect answer patterns were identified including one consistent with this Cartesian label orthogonality confusion (13% pre, 16% post), as well as apparent confusion between projections and operator eigenvalues as indicated by answers involving $\hbar$ or $\hbar/2$ (20% both pre and post). C. Singh noted in her 2001 paper [28] that students sometimes confuse probability amplitudes (as resulting from the projections in this item) with operator eigenvalues. It is possible that this played a role in the instances of overall prefactors involving ½ and/or $\hbar$, but we did not collect sufficient interview evidence to make this claim. It is also possible that the convention sometimes used in higher spin states (see for example [1]) of labeling eigenvectors using the eigenvalue (as in $|sm\rangle$ or $|m =$ $-3\hbar/2\rangle$) could contribute to confusion with prefactors involving $\hbar$. Note that these considerations cannot account for many errors in Item 1,

Many students confuse the addition of two spins with the addition of the projections of those spins onto the z axis. Item 3 was written to quantify this. The most common error was claiming that a spin-up and spin-down particle could only yield a spin-0 system (29% pre, 14% post). A further indication of conflation of terms "spin" and "spin projection" is that several students included "-1" as a possible outcome for the total spin of the system (13% pre, 17% post).

In item 4, the prevalence of specific errors is more salient than the overall scores. We reiterate that in all years that this item was used, exchange symmetry was explicitly addressed in class, on at least one homework assignment, and (at one institution) in optional group work sessions. The treatment of exchange symmetry was similar to (if not identical to) that in Chapter 10 of Shankar's Principles of Quantum Mechanics [1].

Statements (a), (b), and (e) require careful consideration and perhaps the use of a Clebsch-Gordan table (provided), and may require several steps in reasoning. Statements (c),

(d), and (f), however, are more like axioms that 20-40% of student believe to be true when in fact they are not. These latter three errors were very surprising to instructors. The assertion by almost 40% of students (on the posttest) that all bosons must always be in a spin state that is symmetric under exchange, is especially noteworthy.
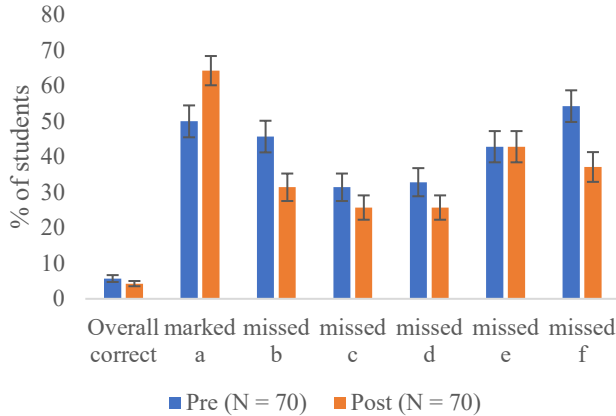

Figure 1: Response proportions for Item 4 (on exchange symmetry). Note that errors are not mutually exclusive. Error bars indicate binomial standard error; note that within-student error on the pre and posttest is correlated.

IV. Conclusions

The properties of spin discussed in this paper are taught in graduate-level quantum mechanics courses across the United States and beyond. This work suggests that many graduate students, even after instruction, do not have basic understanding of fundamental concepts of quantum-mechanics applied to spin-½ states, such as orthogonality, spin addition, and exchange symmetry. This is especially worth attention given that items 1-3 studied here might be considered basic questions in undergraduate quantum mechanics courses. Although the primary goal of this study was to identify student difficulties with spin that are common at the graduate level, we note that they are also persistent: in virtually all cases, instruction seems to have had little to no effect. This is clearly not due to ceiling effects, since none of the post-test item averages rose above 75%.

Some areas of student difficulty discussed in this work had already been identified in undergraduate populations. These areas include stating that quantities labeled with Cartesian coordinates "x", "y", or "z" (such as $|+\rangle_x$ and $|+\rangle_z$) are orthogonal to each other, and difficulty adding spin vectors [8]. The value of these observations is not limited to the confirmatory, due to the tremendous selection effects at work between undergraduate and graduate populations. Other student difficulties were apparently newly identified

in this graduate population. These include a number of misunderstandings of exchange symmetry.

After assessments were completed and scored, results were discussed with instructors. Reactions were noted, and often fell into one of two categories. In many cases, instructors recognized a difficulty as something they had seen before. But surprise was also a common reaction.

The student answers to the questions investigated in this study indicate that the student difficulties in these topics may not be most productively described in terms of stable, coherent misconceptions, but rather in terms of other models of understanding or student answering, such as a resources model [29] or a dual process model [30,31]. A simple example of this is the observed inconsistency between student answers to Items 1 and 2, both of which require the concept of orthogonality but in slightly different contexts and representations. Item 1 explicitly uses the term "orthogonal" which may, in terms of a resources model, naturally evoke a graduate student's substantial resources of Cartesian coordinate systems. Or in terms of a dual process model, rapid, highly accessible associations cued by the term "orthogonal" dominate the decision process. For words like "orthogonal" that students have used for years outside of the Hilbert space context, this cuing could be consistent with the persistence of the student errors from pretest to posttest, even if cuing is different between items. An exploration of student answering from these perspectives is worth further study. Some reliance on existing resources could be shifted by using different phrasing in the question stems, such as asking which states "have a projection of zero" onto the eigenstate in question, rather than invoking the term "orthogonality". Alternatively, one could phrase Item 1 entirely in terms of probabilities of subsequent measurements. Whatever tuning of resources is undertaken would almost certainly need to be informed by a think-aloud interview, or similar process.

Because of the scarcity of research in student understanding of spin at the graduate level, this initial inquiry has been primarily devoted to the identification of what student misunderstandings exist in the graduate population, and which may persist through to the end of graduate level instruction. Clearly, there is more work to be done to better understand the significant difficulties student have, for example, with the rich and complex topic of exchange symmetry.

[1] R. Shankar, *Principles of Quantum Mechanics 2nd Ed.*, (Springer, New York, NY), 1994.

[2] J. Sakurai and Jim Napolitano, *Modern Quantum Mechanics 2nd Ed. Revised*, (Cambridge University Press, Cambridge, United Kingdom), 2018.

[3] I. Zutic, J. Fabian, and S. Das Sarma, Spintronics: Fundamentals and applications, Rev. Mod. Phys. 76 2, 323-410 (2004)

[4] Y. Ren, Z. Qiao, and Q. Niu, Topological phases in two-dimensional materials: a review, Rep. on Prog. In Phys., 79 (6) 066501 (2016).

[5] D. H. McIntyre, C. A. Manogue, and J. Tate, Quantum Mechanics: A Paradigms Approach, 1st ed. (Pearson Education, Inc., San Francisco, CA, 2012)

[6] H. Sadaghiani and J. Munteanu, Spin First instructional approach to teaching quantum mechanics in sophomore level modern physics courses, PERC Proc., 287-290 (2015).

[7] H. Sadaghiani, Spin First vs. Position First instructional approaches to teaching introductory quantum mechanics, PERC Proc., Sacramento, CA, 292-295 (2016).

[8] E. Marshman and C. Singh, Review of student difficulties in upper-level quantum mechanics, Phys. Rev. ST Phys. Educ. Res., **11**, 020117 (2015).

[9] P. Jolly, D. Zollman, S. Rebello, and A. Dimitrova, Visualizing motion in potential wells, Am. J. Phys. **66** (1), 57–63 (1998).

[10] L. Bao, P. Jolly, and E. F. Redish, Student Difficulties with Quantum Mechanics, AAPT Announcer 26 (2), **70-71** (July 1996).

[11] M. C. Wittmann, R. N. Steinberg, and E. F. Redish, Investigating student understanding of quantum physics: spontaneous models of conductivity, Am. J. Phys. **70**, 218 (2002).

[12] S. Goldhaber, S. Pollock, M. Dubson, P. Beale, and K. Perkins, Transforming upper-division quantum mechanics: learning goals and assessment, AIP Conf. Proc. **1179**, 145 (2009).

[13] L. Bao and E. F. Redish, Understanding probabilistic interpretations of physical systems: A prerequisite to learning quantum physics, Am. J. of Phys. **70**, 210 (2002).

[14] S. B. McKagan, K. K. Perkins, and M. Dubson, Developing and researching PhET simulations for teaching quantum mechanics, Am. J. Phys. **76**, 406 (2008).

[15] C. D. Porter and A. F. Heckler, Effectiveness of guided group work in graduate level quantum mechanics, Phys. Rev. Phys. Educ. Res. **16**, 020127 (2020).

[16] B. Brown and C. Singh, Development and evaluation of a quantum interactive learning tutorial on Larmor precession of spin, PERC Proc. Minneapolis, MN, 47-50, (2014).

[17] C. Singh, Student understanding of quantum mechanics at the beginning of graduate instruction, Am. J. Phys. **76**, 277 (2008).

[18] L.D. Carr and S. B. McKagan, Graduate quantum mechanics reform, Am. J. Phys. 77, 308 (2009).

[19] C. Keebaugh, E. Marshman, and C. Singh, Investigating and addressing student difficulties with a good basis for finding perturbative corrections in the context of degenerate perturbation theory, Eur. J. Phys., **39**(5) 055701 (2018).

[20] C. D. Porter, A. Bogdan, and A. Heckler, Student understanding of potential, wavefunctions, and the Jacobian in hydrogen in graduate quantum mechanics, PER Conf. Proc., Sacramento CA, p.244-247 (2016).

[21] C. D. Porter, and A. Heckler, Graduate student misunderstandings in an asymmetric well, Phys. Rev. Phys. Educ. Res., **15**, 010139 (2019).

[22] G. Zhu and C. Singh, Improving students' understanding of quantum mechanics via the Stern–Gerlach experiment, Am. J. Phys., **79**, 499 (2011).

[23] H. Sadaghiani and S. Pollock, "Quantum Mechanics Concept Assessment: Development and Validation Study", Phys. Rev. ST Phys. Educ. Res. **11** (1), 010110 (2014).

[24] E. Marshman and C. Singh, Validation and administration of a conceptual survey on the formalism and postulates of quantum mechanics, Phys. Rev. PER, **15**, 020128 (2019).

[25] E. Marshman and C. Singh, Investigating student difficulties with time dependence of expectation values in quantum mechanics, presented at the Physics Education Research Conference, 2013, Portland, OR (2013).

[26] P. J. Emigh, G. Passante, and P. S. Shaffer, Student understanding of time dependence in quantum mechanics, Phys. Rev. ST Phys. Educ. Res. 11, 020112 (2015).

[27] M. T. H. Chi, ''Thinking Aloud,'' in The Think Aloud Method, edited by M. W. van Someren, Y. F. Barnard, and J. A. C. Sandberg ~Academic, London, 1994, Chap. 1

[28] C. Singh, Student understanding of quantum mechanics, Am. J. Phys. 69(8), 885-896, 2001.

[29] D. Hammer, A. Elby, R. E. Scherr, & E. F. Redish, "Resources, framing, and transfer," in Transfer of Learning: Research and Perspectives, J. Mestre, ed. (Information Age Publishing, 2004).

[30] A. F. Heckler, The Ubiquitous Patterns of Incorrect Answers to Science Questions: The Role of Automatic, Bottom-up Processes, Psychol. Learn. Motivation, Adv. Res. Theory 55, 227 (2011).

[31] M. Kryjevskaia, M. R. Stetzer, and N. Grosz, Answer first: Applying the heuristic-analytic theory of reasoning to examine student intuitive thinking in the context of physics, Phys. Rev. ST Phys. Educ. Res. 10, 020109 (2014).