1. (15 pts) The figure shows two infinite planes of thickness $t$, whose separation is $2d$. Both are conductors and have a surface charge density of $+\sigma$ and $-\sigma$ as shown in the drawing.

a) In unit vector notation, what is the electric field between the two planes?

- $E$ goes from positive to negative $\Rightarrow$ $+$ $\hat{z}$ direction
- Use Gaussian pillbox

$$\oint E \cdot dA = \frac{\sigma}{\varepsilon_0}$$

$$\vec{E} = \frac{\sigma}{\varepsilon_0} \hat{z}$$

Note that unlike with insulators, you do not find each $E$ separately and add.

b) In unit vector notation, what is the electric field inside the upper plane?

the planes are conductors

$\Rightarrow \vec{E} = 0$

c) What is the potential difference between the lower surface ($z = d$) of the upper plane and the upper surface ($z = -d$) of the lower plane?

This is a uniform (constant) electric field

$\Rightarrow$ can use $\Delta V = E \Delta d$

$$\Delta V = E(2d) = \frac{2\sigma d}{\varepsilon_0}$$
Name: Solution

2. (15 pts) Three charges are located at the corners of an equilateral triangle. Each side of the triangle has a length $d$. A point $P$ is located in the center of the triangle, and the center of a coordinate system is also located there.

a) What is the electric field at point $P$ in unit vector notation?

Together, the +2Q charges will create an $E$ field going straight up (by symmetry)

$$\vec{E}_{2Q} = 2 \left( \frac{2Q}{4\pi \epsilon_0 d^2} \right) \sin 30^\circ \hat{j} = \frac{Q}{2\pi \epsilon_0 d^2} \hat{j}$$

$$\vec{E}_{4Q} = \frac{4Q}{4\pi \epsilon_0 d^2} (-\hat{j}) = -\frac{Q}{\pi \epsilon_0 d^2} \hat{j}$$

$$\vec{E}_P = \vec{E}_{2Q} + \vec{E}_{4Q} = \frac{Q}{\pi \epsilon_0 d^2} \left( \frac{1}{2} - 1 \right) \hat{j} = -\frac{3Q}{2\pi \epsilon_0 d^2} \hat{j}$$

b) If $V = 0$ at infinity, what is the electric potential at point $P$?

Use $V = \frac{Q}{4\pi \epsilon_0 r}$ with $r = \frac{d}{\sqrt{3}}$

$$V = \frac{+2Q}{4\pi \epsilon_0 \frac{d}{\sqrt{3}}} + \frac{2Q}{4\pi \epsilon_0 \frac{d}{\sqrt{3}}} + \frac{Q}{4\pi \epsilon_0 \frac{d}{\sqrt{3}}}$$

$$= \frac{3Q}{\pi \epsilon_0 d} = \frac{2\sqrt{3}Q}{\pi \epsilon_0 d}$$

(c) How much energy was required to assemble the three charges from infinity?

For a pair of point charges $U = \frac{Q_1 Q_2}{4\pi \epsilon_0 r}$ (from $U = qV$)

Add up all 3 possible pairs

$$U = \frac{(2Q)(2Q)}{4\pi \epsilon_0 d} + \frac{(2Q)(2Q)}{4\pi \epsilon_0 d} + \frac{(2Q)(2Q)}{4\pi \epsilon_0 d}$$

$$= \frac{4Q^2}{\pi \epsilon_0 d} + \frac{4Q^2}{\pi \epsilon_0 d} + \frac{4Q^2}{\pi \epsilon_0 d}$$

$$= \frac{12Q^2}{\pi \epsilon_0 d}$$

$$= \frac{5Q^2}{\pi \epsilon_0 d}$$
3. (15 pts) The figure shows the end view of a long rod of radius $a$, which is surrounded by a thin concentric cylinder of radius $b$. If both are insulators, the inner rod has a surface charge density of $+\sigma$, and the outer cylinder a surface charge density of $-2\sigma$ on its inner surface.

a) What is the electric field between the rod and cylinder?

We "choose" a Gaussian cylinder of radius $r$.

\[
\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\varepsilon_0}
\]

\[
= \varepsilon_0 \frac{\sigma}{(a - r) l} \cdot r
\]

\[
\Rightarrow E = \frac{\sigma}{\varepsilon_0} \frac{a}{r}
\]

\[\text{(Note: } \frac{\sigma}{(a - r) l} = \frac{2\sigma}{\pi a} \text{)}\]

b) What is the electric field outside the cylinder?

Now the Gaussian cylinder encloses both of the cylinders.

\[
\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\varepsilon_0}
\]

\[
= \varepsilon_0 \frac{\sigma}{(a - r) l} \cdot r - \varepsilon_0 \frac{-2\sigma}{(2\pi a) l} \cdot (2\pi r l) + \varepsilon_0 \frac{(\sigma) (2\pi a l)}{(2\pi a) l}
\]

\[
\Rightarrow E = \frac{\sigma}{\varepsilon_0} \left( \frac{a - 2b}{r} \right)
\]

\[\text{(Note: } a > b)\]

\[\text{c) What is the electric field inside the rod?}\]

The e.f. field in this case is zero since \( q_{\text{enc}} = 0 \Rightarrow \oint \mathbf{E} \cdot d\mathbf{A} = 0 \Rightarrow E = 0 \)
4. (15 pts) Consider the two conducting infinite sheets shown in
the drawing. If the total charge densities (both surfaces) are as
indicated, surfaces "2" and "3" are separated by a distance \( d \), and
the charge density on the right side of the right sheet (surface 4) is
\( +\sigma \).

\[ \oint E \cdot dA = \frac{q_{\text{enc}}}{\varepsilon_0} \]
\[ E \cdot A = \frac{\sigma A}{\varepsilon_0} \]
\[ E = \frac{\sigma}{\varepsilon_0} \]

Total charge per unit area
\( +3\sigma \)
\( -2\sigma \)

b) What is the magnitude of the electric field between surfaces 2 and 3?
\[ \oint E \cdot dA = \frac{q_{\text{enc}}}{\varepsilon_0} \]
\[ q_{\text{enc}} = \frac{3\sigma A}{\varepsilon_0} \]
\[ E \cdot A = \frac{3\sigma A}{\varepsilon_0} \]
\[ E = \frac{3\sigma}{\varepsilon_0} \]

c) What is the potential between surfaces 1 and 4?
The potential difference btw. 1+2, and 3+4 is zero since the sheets are
conductors. Therefore the potential btw. 1+4 = potential btw. 2+3
\[ \Delta V = -\int E \cdot ds = -\int \frac{3\sigma}{\varepsilon_0} ds = -\frac{3\sigma}{\varepsilon_0} d \]

d) What is the charge density on surface 1?
\( \sigma \) (see diagram)
5. (15 pts) The figure shows three concentric conducting thin spherical shells of radii \(a\), \(b\), and \(c\); with total charge of \(-7Q\), \(+5Q\), and \(+3Q\) on each shell, respectively.

5a) What is the charge on the outer surface of the shell with radius \(c\)?

Net charge on \(a + b\) is \(-7Q + 5Q = -2Q\)

\(\Rightarrow\) Charge on inner surface of \(c\) is \(-2Q\)

Charge on outer surface of \(c\) is \(q_{\text{out}} = q_{b} + q_{\text{in}} = +3Q - +2Q = +Q\)

5b) What is the electric potential difference between \(b\) and \(c\)?

Outside a given sphere, can treat it like a point charge at \(c\) (assume just outside \(c\))

\(V_{c} = \frac{-7Q}{4\pi\epsilon_{0}b} + \frac{5Q}{4\pi\epsilon_{0}b} + \frac{3Q}{4\pi\epsilon_{0}c}\)

\(\text{where } c = c\)

\(V_{b} = \frac{-7Q}{4\pi\epsilon_{0}b} + \frac{5Q}{4\pi\epsilon_{0}b} + \frac{3Q}{4\pi\epsilon_{0}c}\)

Potential due to \(c\) is constant everywhere inside \(c\)

\(V_{b} - V_{c} = \frac{2Q}{4\pi\epsilon_{0}b} + \frac{2Q}{4\pi\epsilon_{0}c} = \frac{2Q}{4\pi\epsilon_{0}} \left( \frac{1}{c} - \frac{1}{b} \right)\)

5c) What is the net flux through a Gaussian surface located between \(b\) and \(c\)?

\(\varepsilon_{0} \Phi = q_{\text{enc}}\)

\(\Phi = \frac{q_{\text{enc}}}{\varepsilon_{0}}\)

\(= \frac{q_{a} + q_{b}}{\varepsilon_{0}}\)

\(\Phi = -\frac{2Q}{\varepsilon_{0}}\)
6. (18 pts) A sphere of radius R and volume charge density +\( \rho \) sits atop an infinite plane of surface charge density +\( \sigma \) as shown in the figure.

\[
q = \rho V = \sigma A
\]

\[
V = \frac{4}{3} \pi R^3
\]

\[
E = \frac{\rho R}{\varepsilon_0} + \frac{\sigma}{2\varepsilon_0}
\]

\[
E_{\text{plane}} = \frac{\sigma}{2\varepsilon_0}
\]

\[
\frac{\partial E}{\partial R} = \frac{\rho R}{\varepsilon_0}
\]

\[
\frac{\partial E}{\partial R} = \frac{\rho R}{\varepsilon_0}
\]

\[
E_{\text{plane}} = \frac{\sigma}{2\varepsilon_0}
\]

\[
\text{a) What is the electric field at point } P \text{ on the top of the sphere?}
\]

\[
\vec{E} = E_{\text{sphere}} + E_{\text{plane}}
\]

\[
E_{\text{sphere}} = \frac{1}{4\pi\varepsilon_0} \frac{\rho V}{R^2}
\]

\[
E_{\text{plane}} = \frac{\sigma}{2\varepsilon_0}
\]

\[
\text{b) How much work would be required to raise the sphere by a distance } R?
\]

\[
W = -\sigma V Q = -\sigma \left[ (V_{\text{sphere, f}} + V_{\text{plane, f}}) - (V_{\text{sphere, i}} + V_{\text{plane, i}}) \right]
\]

\[
V_{\text{sphere}} = V_{\text{plane}} = \frac{\sigma R}{2\varepsilon_0}
\]

\[
W = \frac{\sigma R^2}{2\varepsilon_0} \frac{\theta_{\text{sphere}}}{2\varepsilon_0} = \frac{\sigma R^2}{2\varepsilon_0} \frac{4}{3} \pi R^3 = \frac{2\sigma\pi R^4}{3\varepsilon_0}
\]