Section 1 - short problems

Consider the system of three charges shown in the figure:

\[
\begin{aligned}
\vec{E} &= \sum \frac{1}{4\pi\varepsilon_0} \frac{Q_i}{r_i^2} \hat{r}_i \\
&= \frac{1}{4\pi\varepsilon_0} \left[ \frac{+Q}{d^2} + \frac{-3Q}{2d^2} \left( \frac{\hat{e}_x^2}{d^2} \right) + \frac{4Q}{d^2} \right] \\
&= \frac{1}{4\pi\varepsilon_0} \left[ \frac{4Q - 3N_2Q(\hat{e}_x^2)}{d^2} \right] \\
&= \frac{Q}{4\pi\varepsilon_0 d^2} \left[ (4 - 3N_2)\hat{e}_x + (16 - 3N_2)\hat{e}_y \right]
\end{aligned}
\]

I.1 (10 pts): What is the electric potential at point "P"?

\[
V = \frac{1}{\varepsilon_0} \sum \frac{Q_i}{r_i} = \frac{1}{4\pi\varepsilon_0} \left[ \frac{+Q}{d} + \frac{-3Q}{2d} \left( \frac{1}{d^2} \right) + \frac{4Q}{d} \right] = \frac{1}{4\pi\varepsilon_0} \left[ \frac{+2Q - 3N_2Q + 8Q}{2d} \right]
\]

I.2 (15 pts): What is the electric field at point "P" in unit vector notation?

I.3 (10 pts): How much energy was required to assemble this charge distribution from charges originally separated at infinity?

\[
U \rightarrow \frac{+Q(-3Q)}{d} + \frac{(+Q)(+4Q) + (-3Q)(+4Q)}{N_2 d} = \frac{Q}{2d} \left[ -6 + \frac{+N_2 - 2Q}{d} \right]
\]
I.4 (10 pts): As shown in the figure, a straight section of wire carries a current "i" to the right. If the magnetic field is vertical as shown in the figure, draw on the diagram a vector [a circle with an x in it to indicate a vector into the paper, a circle with a dot in it to indicate a vector out of the paper] which shows the direction of the force.

\[ \mathbf{F} = i \mathbf{L} \times \mathbf{B} \]

For the next 3 questions, consider the circuit below:

I.5 (10 pts): How much current flows out of the 20 V battery?

\[ i_1 = \frac{20V - 10V}{20\Omega} = 0.5 \text{ A} \]

I.6 (10 pts): How much power is dissipated by the 7 Ω resistor?

\[ P = i^2 R = (0.5 \text{ A})^2 \times 7\Omega = 1.75 \text{ W} \]

I.7 (12 pts): How much power is supplied by the 10 V battery?

\[ i_2 = \frac{10V}{4\Omega} = 2.5 \text{ A} \]

\[ i_1 + i_3 = i_2 ; \quad i_3 = i_2 - i_1 = 2.5 \text{ A} - 0.5 \text{ A} = 2 \text{ A} \]

\[ P = (10V)(i_3) = (10V)(2\text{ A}) = 20 \text{ W} \]
I.8 (10 pts): As shown in the diagram, two long wires lie along the x-axis and y-axis, respectively, and each carries a current "i" in the positive direction. They do not touch at the crossing point. At \((x, y) = (a, b)\) in the first quadrant, what is the magnetic field in unit vector notation? In this coordinate system, the z-axis is out of the paper.

\[
\mathbf{B}(x=a, y=b) = \frac{\mu_0 i}{2 \pi b} \mathbf{\hat{y}} + \frac{\mu_0 i}{2 \pi a} (-\mathbf{\hat{z}})
\]

\[
= \frac{\mu_0 i}{2 \pi} \left( \frac{1}{b} - \frac{1}{a} \right) \mathbf{\hat{z}}
\]
Part II.

II.1 (25 pts): Consider the current carrying loop shown in the diagram. Each of the curved wires is a 90° section of a circle centered on “R” and the straight wires are radial to “R”. Use the Biot-Savart Law to calculate \( \mathbf{B} \) at “R”. Clearly indicate its direction.

\[
\mathbf{d} \mathbf{B} = \frac{\mu_0 i}{4\pi} \frac{d \mathbf{s} \times \mathbf{r}}{r^3}
\]

For curved segment at \( r = a \)

\[
d \mathbf{s} \times \mathbf{r} \Rightarrow \text{upward vector} \Rightarrow \text{magn. of} \quad r \, ds
\]

\[
d B_a = \frac{\mu_0 i}{4\pi} \frac{ds}{r^2}
\]

\[
B_a = \int d \mathbf{B} = \frac{\mu_0 i}{4\pi a^2} \int ds = \frac{\mu_0 i}{4\pi a^2} \frac{\pi a^2}{2}
\]

\[
B_a = \frac{\mu_0 i}{8a} \quad (\text{upward})
\]

Similarly, for segment at \( r = b \)

\[
B_b = \frac{\mu_0 i}{8b} \quad (\text{downward})
\]

For both straight segments \( d \mathbf{s} \times \mathbf{r} = 0 \)

\[
\Rightarrow B_c = 0
\]

Net \( \mathbf{B} \)

\[
\mathbf{B} = \frac{\mu_0 i}{8} \left( \frac{1}{a} - \frac{1}{b} \right) \quad \text{vector up, out of paper}
\]
II.2 (25 pts): Consider the conducting loop and resistor shown in the diagram. Perpendicular to the page and directed upward at \( t = 0 \) is a magnetic field given by \( B = [5 - 3t] \, \text{T} \).

(a) Indicate clearly on the diagram the direction of current flow in the loop.
(b) What is the magnitude of the current?
(c) What is the power dissipated by the resistor?

\[ E = -\frac{d\Phi_B}{dt} \]

(A) Because the flux is upward (at least at \( t = 0 \)), but decreasing, the current flows counterclockwise to provide an upward contribution to the total flux (Lenz's law).

\[ |\Phi_B| = 8 \cdot 2 \cdot (5 - 3t) \, \text{m}^2 \, \text{T} = (10 - 6t) \, \text{m}^2 \, \text{T} \]

\[ |E| = \left| \frac{d\Phi_B}{dt} \right| = 6 \, \text{m}^2 \, \text{T/s} = 6 \, \text{V} \]

\[ I = \frac{6 \, \text{V}}{3 \, \Omega} = 2 \, \text{A} \]

\[ P = I^2 R = (2 \, \text{A})^2 \cdot 3 \, \Omega = 12 \, \text{W} \]
II.3 (25 pts): A non-conducting inner sphere of radius "b" has a total charge of $-2Q$ uniformly distributed throughout its volume. This inner sphere is surrounded by a thin conducting outer sphere of radius "a" and total net charge $+4Q$.

(a) What is the total charge on the outer surface of the conducting sphere?
(b) What is the electric field for $r < b$?
(c) What is the electric potential at $r = b$?

(a) Choosing a Gaussian surface inside the thin outer shell

\[
\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} \Rightarrow E = 0
\]

\Rightarrow \text{charge on inside of shell} = +2Q \quad (-2Q + 2Q = 0)

\Rightarrow \text{charge on outside} \Rightarrow \text{since} +2C \quad \text{(conservation of charge)}

(b) \quad \oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} \quad \Rightarrow E = \frac{Q}{4\pi r^2} = \frac{1}{3\varepsilon_0} \left\{ \begin{array}{l}
-2Q = \rho \frac{4}{3} \pi b^3 \quad \rho = \frac{-6Q}{4\pi b^3} \\
E = \frac{-6Q}{12\pi \varepsilon_0 b^3}
\end{array} \right.

(c) \quad V = \frac{1}{4\pi \varepsilon_0} \frac{+4Q}{a} \quad + \frac{1}{4\pi \varepsilon_0} \frac{-2Q}{b} = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{1}{a} - \frac{2}{b} \right]$

\text{due to charge on outer shell} \quad \text{due to charge on inner sphere}
II.4 (25 pts): A long, hollow conducting cylinder of inner radius "a" and outer radius "b" carries a current $2I_0$ into the paper. Inside the cylinder a long, thin wire carries a current $5I_0$ out of the paper.

(a) What is the magnetic field for $r > b$?
(b) What is the current density $J$ inside the conductor?
(c) What is the magnitude of the field for $a < r < b$?
(d) What is the force on the long thin wire?

\[ a) \quad \oint \overrightarrow{B} \cdot d\overrightarrow{s} = \mu_0 i_{env} \]
\[ B \cdot 2\pi r = \mu_0 \left( 5I_0 - 2I_0 \right) \]
\[ B = \frac{3 \mu_0 I_0}{2\pi r} \]

\[ b) \quad J = \frac{2I_0}{\pi b^2 \pi a^2} \]
\[ = \frac{2I_0}{\pi (b^2 - a^2)} \]

\[ c) \quad \oint \overrightarrow{B} \cdot d\overrightarrow{s} = \mu_0 i_{env} \]
\[ B \cdot 2\pi r = \mu_0 \left[ \frac{2I_0 \pi (v^2 a^2)^2}{\pi (b^2 - a^2)} - 5I_0 \right] \]
\[ B = \frac{\mu_0 I_0}{2\pi r} \left[ \frac{2(v^2 - a^2)}{(b^2 - a^2)} - 5 \right] \]

\[ d) \quad \overrightarrow{E} = j \overrightarrow{C} \times \overrightarrow{B} \]
\[ \text{at the core, } \overrightarrow{B} \text{ due to the core cylinder} = 0 \]
\[ \Rightarrow \overrightarrow{F} = 0 \]