Passive Optical Resonators (i.e. no gain element)

We ask the question: what happens to light inside a passive cavity? In general it all escapes immediately but, depending on the resonator design, there can exist field modes that only decay away slowly.

We define a field mode as one which has the form:

$$\vec{E}(\vec{r},t) = E_0 \underbrace{\vec{u}(\vec{r})\, e^{i\omega t}}_{\text{standing wave}}\, e^{-t/2\tau_c}$$

"c" for cavity

polarization is usually important

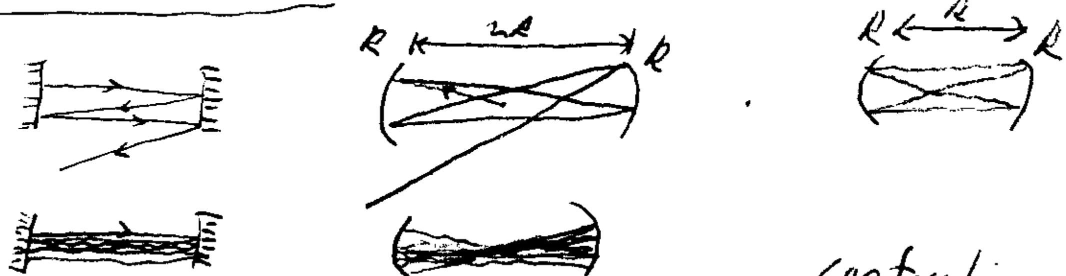
standing wave: time and space dependence separates

exponential decay

I decays as $e^{-t/\tau}$

This is not the most general def'n but it is more than adequate to consider the main practical question before us: how does one build an optical cavity that can store light $\Leftrightarrow$ that is sufficiently stable that a photon will survive for a sufficient number of round trips

Ray optics picture (see "ray trajectory applet")



plane-parallel   concentric   confocal ← 3 main limiting cases

You can study stability, but it's not obvious what the spatial mode looks like.

- A cavity that has stable modes (fields as defined above with $\tau_c \gg \tau_{RT} = \frac{2L}{c}$) is called a resonator.
    - stores energy (Just as a resonant circuit must be able to do)
    - has resonant frequencies. These can be non-trivial, but as a simple example:

To have a standing wave, we require

$$n\left(\frac{\lambda}{2}\right) = L$$

$$\lambda = \frac{2L}{n}$$

$$\nu_n = n\frac{c}{2L}$$

For $L = 1\,\text{m}$, $\Delta\nu = 150$ MHz

$\nu_{opt} \approx 10^{14}$ Hz so n is large.

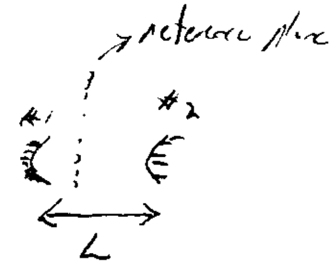
$$\nu_n - \nu_{n-1} = \frac{c}{2L} \equiv \Delta\nu = \text{mode spacing}$$

Lasers for precision spectroscopy might have only 1 resonant mode active,

Short pulse lasers will have 100's present.

- In general τ_c depends on the polarization and on $\vec{u}(r)$. You don't want $\tau_c \to \infty$. You get useful laser light by letting light leak out of the cavity. (Although some experiments are done inside the laser cavity.)

Photon Lifetime and Cavity Q (5.3)



We consider a 2-mirror cavity.

Let: R_1 = reflectivity of mirror 1
R_2 = " " " 2
T_i = internal loss per pass
due to * diffraction (eg: light "missing" the mirrors)
* scattering of light out of the cavity
(eg: because of imperfect AR coatings)
* loss elements placed inside the cavity (eg: filters)

I_1 = intensity after 1 round trip

$$= I_0 R_1 R_2 (1-T_i)^2 \longrightarrow \text{1 Round trip = 2 passes}$$

$$I_n = \left[R_1 R_2 (1-T_i)^2\right]^n I_0 \qquad \text{true for any reference plane}$$

F = photon flux = $\frac{I}{h\nu}$

ϕ = # photons in the cavity = $\left(\int F\, dA\right) \tau_{RT} \propto I$ so long as the spatial mode is constant

$$\phi_n = \left[R_1 R_2 (1-T_i)^2\right]^n \phi_0$$

$$\phi(t=nT_R) = \left[R_1 R_2 (1-T_i)^2\right]^n \phi(t=0)$$

We wish to write this:

$$\phi(nT_R) = e^{-nt_R/t_c}\, \phi(t=0)$$

$$I(nT_R) = [\;]^n I_0 = e^{-nT_R/\tau_c} I_0$$

$$e^{-nT_R/\tau_c} = \left[R_1 R_2 (1-T_i)^2\right]^n$$

$$\tau_c = -\frac{2L}{c \ln\left[R_1 R_2 (1-T_i)^2\right]}$$

→ If [ ] is close to 1 (R_1, R_2 close to 1, T_i small)
τ_c is large and
e^{-t_R/t_c} is close to 1
and
$\phi(t) = e^{-t/\tau_c} \phi_0$ ✓

For a commercial high power laser (Nd:YAG, Ar$^+$, ...)

we might have

$$\left.\begin{array}{l} R_1 = 0.9 \\ R_2 = 1 \\ T_i = 0.03 \end{array}\right\} \quad \ln[\;] = -0.166$$

$L = 1.5$ m $\qquad \tau_R = \frac{2L}{c} = 10$ ns $\quad$ ($\Delta\nu = 100$ MHz

$\tau_c = 60$ ns ✓ $\quad$ (1 m pass through the gain medium)

If $T_i = 0 \qquad \tau_c = 94$ ns

If $R_1 = 1 \qquad \tau_c = 164$ ns

Limiting the internal loss to 3% per pass is not easy.

Now, we have $E(t) \propto e^{i\omega t} e^{-t/2\tau_c}$

Exponential decay yields a Lorentzian lineshape.

The spectral intensity FWHM $\boxed{\Delta\nu_c = \frac{1}{\pi \tau_c}}$

The text reports $\Delta\nu_c = \frac{1}{2\pi\tau_c}$ but no one uses HWHF.

For the example above, this yields $\Delta\nu_c = 2.7$ MHz

extremely narrow compared to $\nu_{optical} \approx 10^{14}$ Hz.

$I(\nu) \propto \frac{1}{1+(2\pi\tau_c)^2(\nu-\nu_0)^2}$

What does this mean?

Same thing really as it did when we considered radiative decay.

The cavity can support an off-resonance frequency for a finite period of time.

For any resonant system we define

$$Q \equiv 2\pi \times \frac{\text{energy stored}}{\text{energy lost in 1 cycle}}$$

For a cavity: Work with Φ = total # photons in cavity $\propto I$

Energy stored = $\Phi h\nu$ $\qquad$ 1 optical period $\qquad \Phi = \Phi_0 e^{-t/\tau_c}$

$$\text{Energy lost in one cycle} = -\frac{d(\Phi h\nu)}{dt}\left(\frac{1}{\nu}\right) = -\frac{h\nu}{\nu}\left(-\frac{1}{\tau_c}\right)\Phi$$

$$Q = 2\pi \tau_c \nu = 2\frac{\nu}{\Delta\nu_c}$$

Text has $Q = \dfrac{\nu}{\Delta\nu_c}$

Again, this would only be true if $\Delta\nu_c$ was $\approx$ 1MHz

For 1064 nm light (~~[illegible]~~ Nd:YAG) our example yields

$$Q = 2\frac{c}{\lambda\Delta\nu_c} = 1.0\cdot10^{8} \checkmark$$

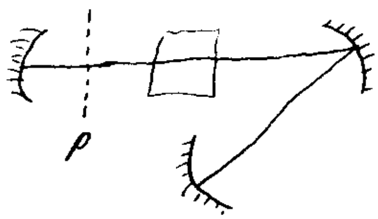
High Q $\Rightarrow$ low loss

"Q-switch" refers to the Q we've just defined.

Stability (5.4, 5.5) [Using Yariv's treatment section 7.2]

For a cavity described by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with reference plane p:

roundtrip



What is required for a Gaussian beam to be a cavity mode?

⇨ Let's neglect the effect of finite sized optics. (Assume $w_0 \ll D_{optic}$)

The overall intensity can change, but not the spatial mode.

We require:

$$q_{p+RT} = q_p \equiv q \quad \Rightarrow \quad q = \frac{Aq + B}{Cq + D}$$

Solving for $\frac{1}{q}$ yields:

$$\frac{1}{q} = \frac{C + D/q}{A + B/q} \quad \Rightarrow \quad B\left(\frac{1}{q}\right)^2 + (A-D)\frac{1}{q} - C = 0$$

$$\frac{1}{q} = \frac{(D-A) \pm \sqrt{(D-A)^2 + 4BC}}{2B}$$

Because $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ starts and ends in the same place, $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is unitary.

Eliminate C

$$AD - BC = 1 \quad \Rightarrow \quad (D-A)^2 + 4BC = (D-A)^2 + 4AD - 4 = (D+A)^2 - 4$$

$$\frac{1}{q} = \frac{D-A}{2B} \pm \frac{\sqrt{\left(\frac{D+A}{2}\right)^2 - 1}}{B}$$

And finally:

$$\frac{1}{q} = \frac{D-A}{2B} \pm i\,\frac{\sqrt{1-\left(\frac{D+A}{2}\right)^2}}{B}$$

But,

$$\frac{1}{q} = \frac{1}{R} - i\,\frac{\lambda}{\pi w^2 n}$$

For a confined beam w must be real and positive.

So, $\sqrt{}$ must be real, or

★ $$\boxed{\left(\frac{D+A}{2}\right)^2 \le 1}$$ Stability criterion.

★ Moreover, $$R_p = \frac{2B}{D-A}$$

Depending on the sign of B, we use the + or − solution so that $\frac{\sqrt{}}{B} \geq 0$. Alternatively, we can write

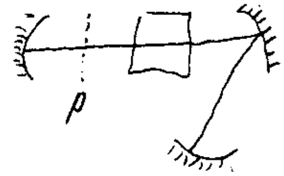
★ $$w_p = \sqrt{\frac{\lambda}{\pi n}}\;\frac{|B|^{1/2}}{\left[1-\left(\frac{D+A}{2}\right)^2\right]^{1/4}}$$

★ And R & w anywhere else can be found by using the ABCD law to propagate the solution.

Stability criterion says w_p must be finite

Stability Condition via Ray Tracing Analysis (SKIP)

For a cavity described by $\begin{bmatrix}A & B\\ C & D\end{bmatrix}$ for the given reference plane P:



Suppose we have ray $\begin{pmatrix} r_0 \\ r_0' \end{pmatrix}$ leaving P. After n round trips:

$$\begin{pmatrix} r_n \\ r_n' \end{pmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}^n \begin{pmatrix} r_0 \\ r_0' \end{pmatrix}$$

Since $\begin{bmatrix}A & B\\ C & D\end{bmatrix}$ starts and ends in the same place it is unitary and:

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}^n = \frac{1}{\sin\theta}\begin{bmatrix} A\sin n\theta - \sin(n-1)\theta & B\sin n\theta \\ C\sin n\theta & D\sin n\theta - \sin(n-1)\theta \end{bmatrix}$$

w/ $\cos\theta = \frac{A+D}{2}$.

If θ is real, $A+B+C+D$ are well behaved. The ray can't diverge arbitrarily far from the optic axis

$\Rightarrow$ the ray is confined and the cavity stable.

(If θ is complex, then $\sin n\theta$ will contain exponentially growing parts.)

$$\left(\sin n\theta = \frac{e^{-nb}e^{ina} + e^{nb}e^{-ina}}{2i} \quad \text{if } \theta = a + ib\right)$$

so, we require $\boxed{-1 \le \frac{A+D}{2} \le 1}$ ✓

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$$\left|\begin{bmatrix}A & B\\ C & D\end{bmatrix}^n\right| = \frac{1}{\sin\theta}\left[AD\sin^2 n\theta + \sin^2(n-1)\theta - \underbrace{(A+D)}_{2}\overset{\sin n\theta}{}\sin(n-1)\theta - BC\sin^2 n\theta\right] =$$

$$= \frac{1}{\sin^2\theta}\left[\sin^2 n\theta + \sin^2(n-1)\theta - 2\underbrace{\cos\theta\sin n\theta\sin(n-1)\theta}_{\gamma}\right]$$

$$\sin n\theta = \sin((n-1)\theta+\theta) = \sin(n-1)\theta\cos\theta + \cos(n-1)\theta\sin\theta$$

$$\gamma = \sin^2(n-1)\theta\cos^2\theta + \cos\theta\cos(n-1)\theta\sin\theta\sin(n-1)\theta$$

$\boxed{n=1}$

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}^n = \frac{1}{\sin\theta}\begin{bmatrix}A\sin\theta & B\sin\theta\\ C\sin\theta & D\sin\theta\end{bmatrix}$$

$$\left|\begin{bmatrix}A & B\\ C & D\end{bmatrix}^n\right| = \frac{1}{\sin^2\theta}\left[AD\sin^2\theta - BC\sin^2\theta\right] = AD - BC = 1 \checkmark$$

$$\sin A\cos B = \tfrac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\gamma = \sin(n-1)\theta\,\tfrac{1}{2}\left[\sin(n-1)\theta + \sin(n+1)\theta\right]$$

$$[\;] = \sin^2 n\theta - \sin(n-1)\theta\sin(n+1)\theta$$

$$\sin(n\theta+\theta) = \sin n\theta\cos\theta + \sin\theta\cos n\theta$$

$$\sin(n\theta-\theta) = \sin n\theta\cos\theta - \sin\theta\cos n\theta$$

$$\sin(n-1)\theta\sin(n+1)\theta = \sin^2 n\theta\cos^2\theta - \cos^2 n\theta\sin^2\theta$$

$$[\;] = \sin^2 n\theta(1-\cos^2\theta) + \cos^2 n\theta\sin^2\theta$$

$$\frac{[\;]}{\sin^2\theta} = \sin^2 n\theta + \cos^2 n\theta = 1 \checkmark$$