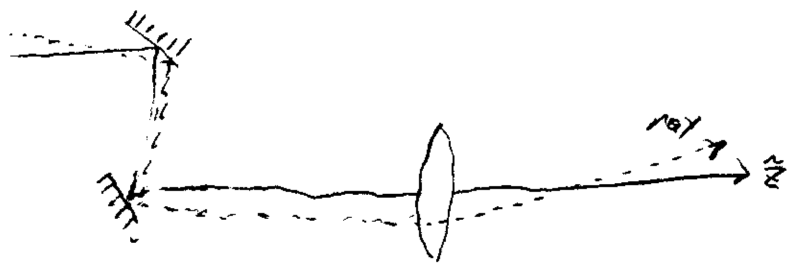
Matrix Method 4.2

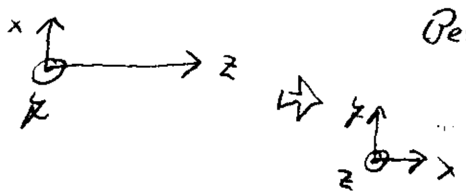
This can be used to describe ray and some aspects of wavefront propagation. We represent a laser beam by a bundle of rays.

The optic axis or z-axis is a path approximately followed by the rays

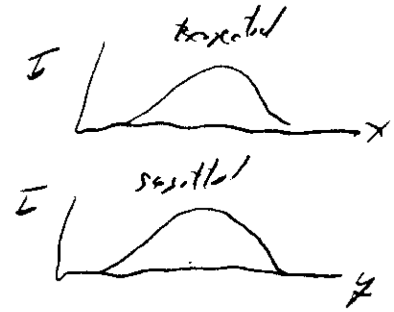


We choose this axis for our convenience.

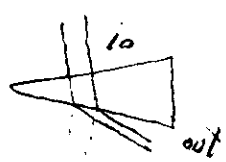
In a diagram like the above, we usually define two reference planes: tangential and sagittal.

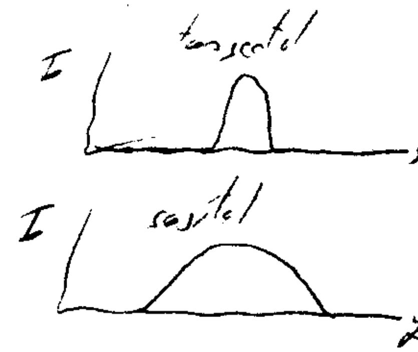


Beam coming out of page before prism

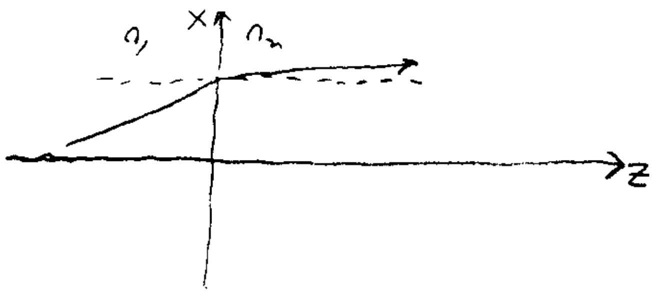


After a prism...





We can describe a ray in the tangential plane by specifying $x(z)$, $x'(z)$ $\quad x' \equiv \frac{dx}{dz}$



and likewise for the sagittal ($y(z)$ & $y'(z)$).
Or we can project an arbitrary ray.

☆ Paraxial ray assumption: angle wrt z-axis, θ, is small so
$\sin\theta \approx \tan\theta \approx \theta$.

With this assumption, we can connect any plane $z = z_1$ with any other plane $z = z_2$.

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{z_2} = \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{z_1}$$

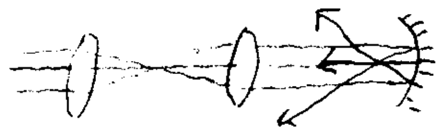
with the "--" independent of x, x', y, y'.

With a particular choice of axes ⇒
some components treat the planes identically : mirror on axis $\Rightarrow$ $\quad \begin{pmatrix} \, \end{pmatrix} = \begin{pmatrix} A & B & \emptyset & \\ C & D & & \\ & \emptyset & A & B \\ & & C & D \end{pmatrix}$

" " " " " differently but orthogonally : mirror off axis $\Rightarrow$ $\quad = \begin{pmatrix} A_1 & B_1 & \emptyset & \\ C_1 & D_1 & & \\ & \emptyset & A_2 & B_2 \\ & & C_2 & D_2 \end{pmatrix}$

and some require the general case.

~~Introduce ABCD~~, Let's just follow propagation
Assume no 1st case.
in a single unspecified plane. For a system like:



all planes are identical and all you need for each ray is: $r(z)$ $\quad r'(z)=\frac{dr}{dz}$

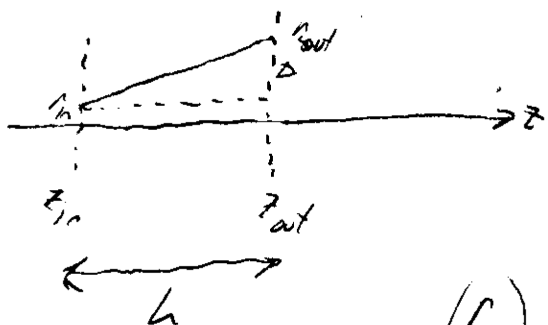
$$\begin{pmatrix} r_{out} \\ r'_{out} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_{in} \\ r'_{in} \end{pmatrix}$$

$r' = \theta$ paraxial approx

$\theta > 0$

Extension back to the general case is straightforward.

#1) $\boxed{\text{Propagation in air } (n_{air}=1)}$ Given $\begin{pmatrix} r_{in} \\ r'_{in} \end{pmatrix}$ calculate $\begin{pmatrix} r_{out} \\ r'_{out} \end{pmatrix}$



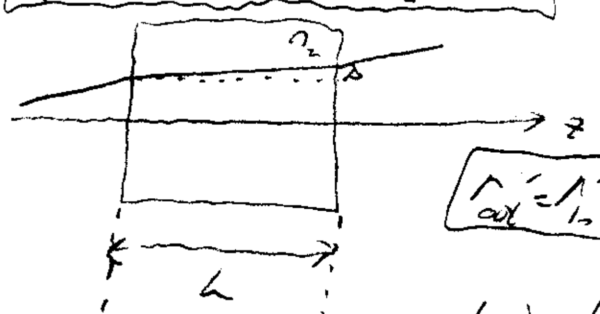
Clearly $r'_{out} = r'_{in}$ ✓

$r_{out} = r_{in} + \Delta = r_{in} + r'_{in}(z_i)\,L$ ✓

$$\begin{pmatrix} r_{out} \\ r'_{out} \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_{in} \\ r'_{in} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

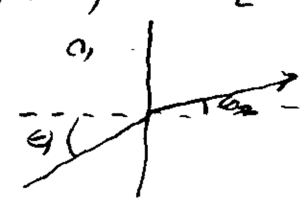
#2) $\boxed{n_1 \to \text{slab index } n_2 \to n_1}$



$\boxed{r'_{out} = r'_{in}}$

careful: these θ's use a different sign

$n_1 \sin\theta_1 = n_2 \sin\theta_2$



$n_1\theta_1 \approx n_2\theta_2$

At z_1 in but before angle change. At z_2 out and after angle change.

$$\begin{pmatrix} r_{out} \\ r'_{out} \end{pmatrix} = \begin{pmatrix} 1 & \frac{n_1}{n_2}L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_{in} \\ r'_{in} \end{pmatrix}$$

$r_{out} = r_{in} + \Delta$

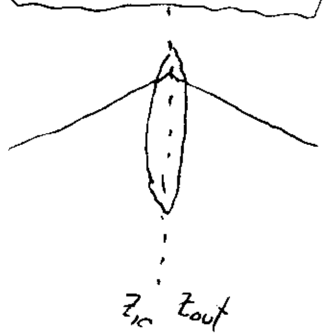
$= r_{in} + \tan\theta_2 L$

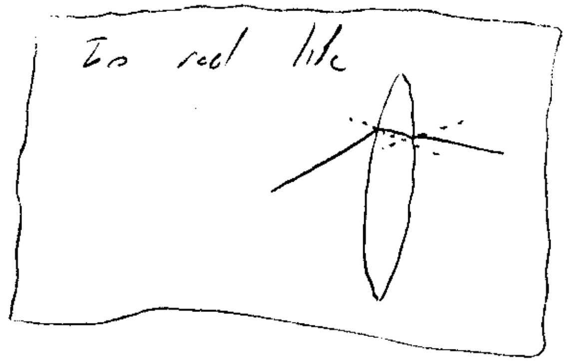
$\tan\theta_2 \approx \theta_2 \approx \frac{n_1}{n_2}\theta_1 \approx \frac{n_1}{n_2} r'_{in}$

$r_{out} = r_{in} + \frac{n_1}{n_2} r'_{in} L$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \frac{n_1}{n_2}L \\ 0 & 1 \end{pmatrix}$$

#7) $\boxed{\text{Thin lens in air}}$ Treat the lens as a "point" interaction (thin lens approx) [4]





okay if paraxial

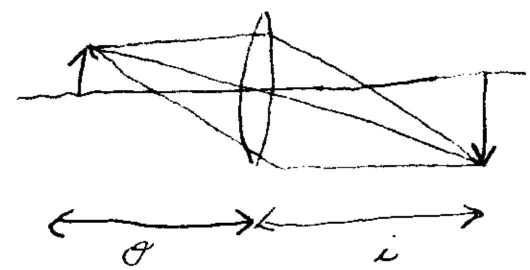
↓

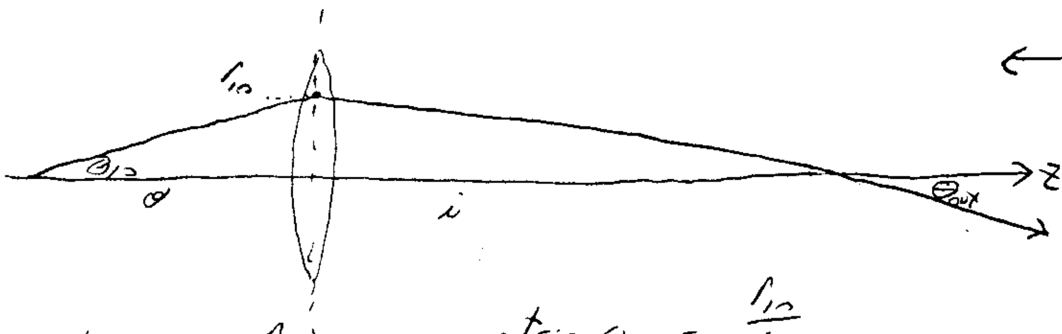
f not too short
beam not too wide

$r_{in} = r_{out}$

Now, we have the thin lens equation,

$$\frac{1}{f} = \frac{1}{o} + \frac{1}{i}$$





$$\tan\theta_{in} = \frac{r_{in}}{o} \qquad -\tan\theta_{out} = \frac{r_{in}}{i}$$

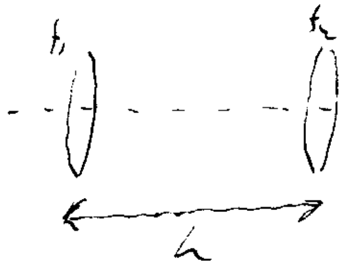
$$r'_{in} = \frac{r_{in}}{o} \qquad -r'_{out} = \frac{r_{in}}{i}$$

$$\frac{1}{f} = \frac{r'_{in}}{r_{in}} - \frac{r'_{out}}{r_{in}}$$

$$r'_{out} = r'_{in} - \frac{r_{in}}{f}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \checkmark$$

Combination: Two lenses separated by L



$$\begin{pmatrix} r_{out} \\ r'_{out} \end{pmatrix} = \begin{pmatrix} \text{lens } \#2 \end{pmatrix} \begin{pmatrix} \text{propagate} \\ \text{in} \\ \text{air} \end{pmatrix} \begin{pmatrix} \text{lens } \#1 \end{pmatrix} \begin{pmatrix} r_{in} \\ r'_{in} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f_1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f_2}+\frac{L}{f_1 f_2}-\frac{1}{f_1} & -\frac{L}{f_2}+1 \end{pmatrix} \quad \checkmark$$

$$\frac{L}{f_1 f_2} - \left(\frac{1}{f_1}+\frac{1}{f_2}\right)$$

$$= \begin{pmatrix} 1-\frac{L}{f_1} & L \\ \frac{L-f_1-f_2}{f_1 f_2} & 1-\frac{L}{f_2} \end{pmatrix} \quad \checkmark$$

Correct?

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC = 1 \Rightarrow AD-BC = \left(1-\frac{L}{f_2}-\frac{L}{f_1}+\frac{L^2}{f_1 f_2}\right) - \left(\frac{L^2}{f_1 f_2}-\frac{L}{f_2}-\frac{L}{f_1}\right) = 1 \checkmark$$

If $L=0 \Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\left(\frac{1}{f_1}+\frac{1}{f_2}\right) & 1 \end{pmatrix} \quad \checkmark$

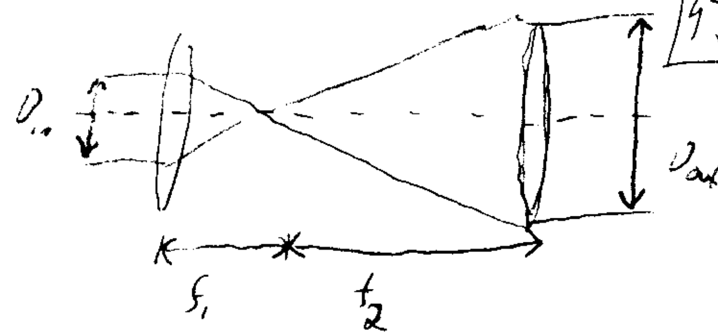
Talk about powers.

If $\Rightarrow$ $f_1 = f$

$f_2 = 3f$

$L = f_1 + f_2 = 4f$

$r'_{in} = 0$



$D_{out} = \frac{f_2}{f_1} D_{in}$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1-4 & 4f \\ 0 & 1-\frac{4}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 4f \\ 0 & -\frac{1}{3} \end{pmatrix} \checkmark$$

$$\begin{pmatrix} r_{out} \\ r'_{out} \end{pmatrix} = \begin{pmatrix} -3 & 4f \\ 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} r_{in} \\ 0 \end{pmatrix} = \begin{pmatrix} -3 r_{in} \\ 0 \end{pmatrix} \checkmark$$ For the ray drawn above.

The real power of this method is that it can be computerized.

Also, it has an especially nice result when applied to a cavity.

$$m = \frac{f}{2\theta_1 - 3f^2}$$

$$\frac{1}{f_1} = \frac{1}{\theta_1} + \frac{1}{i_1}$$

$$i_1 = \left(\frac{1}{f_1} - \frac{1}{\theta_1}\right)^{-1} = \frac{f_1\theta_1}{\theta_1 - f}$$

$$\frac{1}{f_2} = \frac{1}{f_1 + f_2 - i_1} + \frac{1}{i_2}$$

$$\frac{1}{f} = \frac{1}{4f - i_1} + \frac{1}{i_2}$$

$$\frac{1}{i_2} = \frac{1}{f} - \frac{1}{4f - i_1}$$

$$= \frac{3f - i_1}{4f^2 - i_1 f}$$

$$i_2 = \frac{4f^2 - \frac{f^2\theta_1}{\theta_1 - f}}{3f - \frac{f\theta_1}{\theta_1 - f}} = \frac{4f^2\theta_1 - 4f^3 - f^2\theta_1}{3f\theta_1 - 3f^2 - f\theta_1}$$

$$= \frac{3f\theta_1 - 4f^2}{2\theta_1 - 3f^2}$$

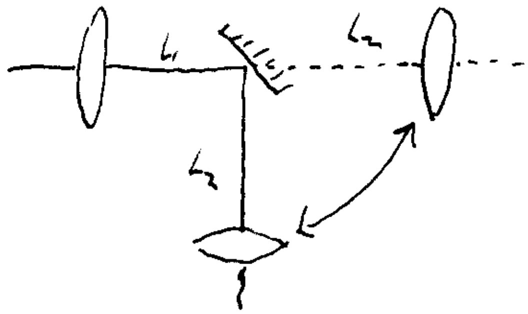
$$m = m_1 m_2 = \left(-\frac{i_1}{\theta_1}\right)\left(-\frac{i_2}{\theta_2}\right)$$

$$= \left(\frac{f}{\theta_1 - f}\right)\left(\frac{3f\theta - 4f^2}{2\theta_1 - 3f^2} \frac{1}{4f - f\theta_1/(\theta_1 - f)}\right)$$

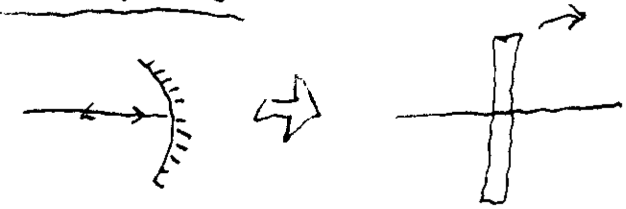
$$= \frac{1}{\theta_1 - f} \frac{3f\theta_1 - 4f^2}{2\theta - 3f^2} \frac{\theta_1 - f}{3\theta - 4f}$$

Flat mirrors

Note that flat mirrors can be ignored.

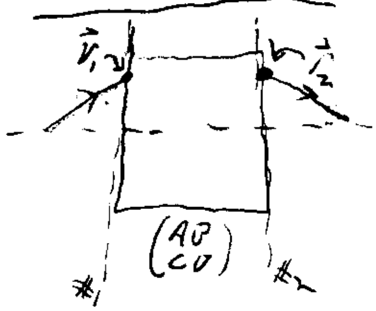


Curved Mirrors

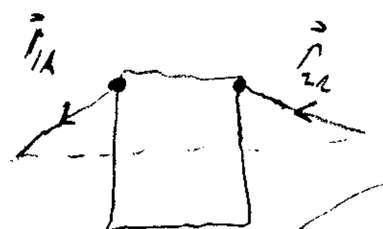


$$\begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$$

Go in reverse



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$



still unitary

$$r_2 = A r_1 + B r_1'$$

$$r_2' = C r_1 + D r_1'$$

$$\begin{pmatrix} r_{1R} \\ r_{1R}' \end{pmatrix} = \begin{pmatrix} A_R & B_R \\ C_R & D_R \end{pmatrix} \begin{pmatrix} r_{2R} \\ r_{2R}' \end{pmatrix}$$

$$\begin{pmatrix} r_1 \\ -r_1' \end{pmatrix} = \begin{pmatrix} A_R & B_R \\ C_R & D_R \end{pmatrix} \begin{pmatrix} r_2 \\ -r_2' \end{pmatrix}$$

$$\begin{pmatrix} r_2 \\ -r_2' \end{pmatrix} = \begin{pmatrix} D_R & -B_R \\ -C_R & A_R \end{pmatrix} \begin{pmatrix} r_1 \\ -r_1' \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = \begin{pmatrix} AD - BC & -AB + BA \\ CD - DC & -BC + DA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\ )_R = \begin{pmatrix} D & B \\ C & A \end{pmatrix}$$

$$r_2 = D_R r_1 + B_R r_1'$$

$$r_2' = C_R r_1 + A_R r_1'$$

$A_R = D$ $\quad B_R = B$

$C_R = C$ $\quad D_R = A$

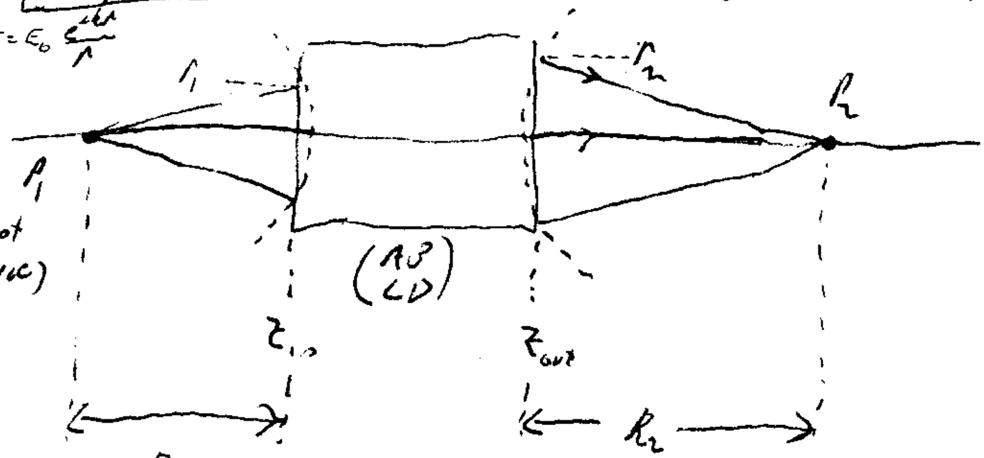
$$\boxed{\begin{pmatrix} r_{1R} \\ r_{1R}' \end{pmatrix} = \begin{pmatrix} D & B \\ C & A \end{pmatrix} \begin{pmatrix} r_{2R} \\ r_{2R}' \end{pmatrix}} \checkmark$$

Spherical waves

① $\boxed{\text{Completely characterized by } R \text{ and wave center}}$

$E = E_0 \frac{e^{ikr}}{r}$

② Assume a spherical wave is still spherical after going through an ABCD system ⇒ valid if system = propagation in air, mirrors, lenses (although "tan" also planes will have diff. radii of curvature if lenses and mirrors are off axis)



$\approx R_1$

(normal approx) (good if $\frac{r_1}{R_1}$ small)

$$\tan\theta_1 \approx \frac{r_1}{R_1} \qquad -\tan\theta_2 = \frac{r_2}{R_2}$$

$$r_1' \approx \frac{r_1}{R_1} \qquad r_2' = -\frac{r_2}{R_2}$$

$$R_1 = \frac{r_1}{r_1'} \qquad R_2 = \frac{r_2}{r_2'}$$

For the case shown, $R_1 > 0$ center of curvature to left of wavefront

$R_2 < 0$ ... to right

$$R_2 = \frac{A r_1 + B r_1'}{C r_1 + D r_1'} = \frac{A \frac{r_1}{r_1'} + B}{C \frac{r_1}{r_1'} + D}$$

$$\boxed{R_2 = \frac{A R_1 + B}{C R_1 + D}} \quad \checkmark$$

Eg) $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ = prop in air for length L $= \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

$$R_2 = R_1 + L$$