# Degenerate Two Level System

**(where each "level" is made up of a number of degenerate sublevels)**

The formalism we've developed can be applied to this system, or more complicated ones, by extending our definition of the cross-section and other terms. With this done we can, in many cases, use the same equation of motion we have been using and determine how light evolves as it passes through an absorbing or gain medium. The complexities of the medium are buried in the cross-section.

I'd like to quickly illustrate how this is done so there is no mystery, hence these notes. However, it is more important that we continue on and cover saturation and ASE in class. (The subject is covered in the text quite well, so this handout is just offered in addition.)

We have:

The lower level, level #1, is taken to have $g_1$ degenerate states, called sublevels, associated with it. The upper level, #2, is taken to have $g_2$ sublevels.

A transition between any one of the upper sublevels and any one of the lower sublevels is possible, although not necessarily with equal strength.

We assume some mechanism couples the sublevels of the upper level to each other so that they always have equal populations, and likewise for the lower level. In other words:

\# atoms per unit volume in any sublevel of the lower level =
(total # atoms per unit volume in the lower level) / $g_1$

$$N_{1i} = \frac{1}{g_1} N_1 \qquad \text{("i" indicates a specific lower sublevel)}$$

and likewise,

$$N_{2j} = \frac{1}{g_2} N_2 \qquad \text{("j" indicates a specific upper sublevel)}$$

The population in the upper state can change due to a transition between any of the upper sublevels and any of the lower sublevels, or due to decay from any of the upper sublevels.

$$\frac{dN_2}{dt} = -\sum_{i=1}^{g_1} \sum_{j=1}^{g_2} \left[ W_{ji} N_{2j} - W_{ij} N_{1i} + \frac{N_{2j}}{\tau_{ji}} \right]$$

Stimulated emission from sublevel j to sublevel i.

Absorption from sublevel i to sublevel j

Decay of sublevel j via spontaneous emission and non-radiative transistions.

Note we have allowed the transition and decays rates to be different for each pair of sublevels. However, for any given pair of states, $W_{ji} = W_{ij}$. Using the equations for $N_{1i}$ and $N_{2j}$ above:

$$\frac{dN_2}{dt} = -\sum_{i=1}^{g_1} \sum_{j=1}^{g_2} \left[ W_{ji} \frac{N_2}{g_2} - W_{ji} \frac{N_1}{g_1} + \frac{N_2}{g_2} \frac{1}{\tau_{ji}} \right]$$

Pulling the constants out of the sum:

$$\frac{dN_2}{dt} = -\left[\frac{N_2}{g_2}\sum_{i,j} W_{ji} - \frac{N_1}{g_1}\sum_{i,j} W_{ji} + \frac{N_2}{g_2}\sum_{i,j}\frac{1}{\tau_{ji}}\right]$$

Let W be the effective single atom transition rate between the upper and lower levels:

$$W \equiv \sum_{i,j} W_{ji}$$

By "effective", I mean a rate that allows us to treat all the upper sublevels as being one big upper level and likewise for the lower levels.

Let τ be the effective time for a decay from the upper level to the lower level:

$$\frac{1}{\tau} = \frac{1}{g_2}\sum_{i,j}\frac{1}{\tau_{ji}}$$

So now we have,

$$\frac{dN_2}{dt} = -W\left(\underbrace{\frac{N_2}{g_2}}_{\text{Stimulated emission}} - \underbrace{\frac{N_1}{g_1}}_{\text{Absorption}}\right) - \underbrace{\frac{N_2}{\tau}}_{\text{Decay via spontaneous emission and non-radiative transistions.}}$$

For each stimulated emission, we increase the number of photons by one. For each absorption we decrease the number by one. The spontaneous decays emit photons in random directions with random phases and do not affect the photon flux of a beam traveling through the medium. Thus, the equation of motion for the flux is:

$$dF = W\left(\frac{N_2}{g_2} - \frac{N_1}{g_1}\right) dz$$

Note that each term comes in with the opposite sign from the equation above.

Now, finally, we define effective cross-sections for making a transition upwards or downwards:

$$\sigma_{12} \equiv \frac{W/g_1}{F}$$

$$\sigma_{21} \equiv \frac{W/g_2}{F}$$

The two-level atom had: $\sigma_{12} = \sigma_{21} = W/F$. For the effective cross-sections, however, we have:

$$g_2\sigma_{21} = g_1\sigma_{12}$$

Now that we have our effective cross-sections, let's introduce effective gain and absorption coefficients.

If $N_2/g_2 > N_1/g_1$, then $dF > 0$: F increases. We have gain.

$$dF = \sigma_{21} g_2 F\left(\frac{N_2}{g_2} - \frac{N_1}{g_1}\right)dz$$

So define the effective gain coefficient as:

$$g \equiv \sigma_{21}\left(N_2 - N_1\frac{g_2}{g_1}\right)$$

Thus,

$$\boxed{dF = gFdz}$$

Similarly, if $N_2/g_2 < N_1/g_1$, $dF < 0$ and we have loss. Define an effective absorption coefficient as:

$$\alpha \equiv \sigma_{12}\left(N_1 - N_2\frac{g_1}{g_2}\right)$$

and we get:

$$\boxed{dF = -\alpha F dz}$$

Typically when light passes through an absorbing medium essentially all the population is in the lower sublevels ($N_1 >> N_2$) and we have:

$$\boxed{\alpha = \sigma_{12} N_1}$$

For a four-level laser (the type of laser most often used in commercial and scientific applications), the population is lower level is very small ($N_2 >> N_1$) and we have:

$$\boxed{g = \sigma_{21} N_2}$$

The boxed equations have the same form as the relations found for the non-degenerate two-level case.