Quiz # 2

NAME ______________________________

An ideal gas consists of a large number N of ^{7}Li atoms confined to a volume V. The atoms have 3 spin states but are otherwise identical fermions. Use the grand canonical ensemble with chemical potential $\varepsilon_F = p_F^2/2m$ and temperature 0 to derive its ground state energy U.

(a) Express the sum over orbitals in terms of integrals over position vectors $\boldsymbol{r} = (x, y, z)$ and momentum vectors $\boldsymbol{p} = (p_x, p_y, p_z)$.
Express the sum over orbitals in terms of an integral over the magnitude p of the momentum vector.

$$\sum_{\text{orbitals}} = 3\int \frac{d^3r\,d^3p}{h^3} = \frac{3V}{h^3}\int_0^\infty 4\pi p^2 dp$$

$$= \frac{12\pi V}{h^3}\int_0^\infty dp\, p^2$$

(b) Express N and U as definite integrals (with upper and lower limits) over p.

$$N = \frac{12\pi V}{h^3}\int_0^{p_F} dp\, p^2$$

$$U = \frac{12\pi V}{h^3}\int_0^{p_F} dp\, p^2 \frac{p^2}{2m}$$

(c) Evaluate the integrals over p.

$$\int_0^{p_F} dp\, p^2 = \frac{p_F^3}{3}$$

$$\int_0^{p_F} dp\, p^4 = \frac{p_F^5}{5}$$

(d) Eliminate the chemical potential to get an expression for U as a function of N and V.

$$N = \frac{12\pi V}{h^3}\frac{p_F^3}{3} \Longrightarrow p_F = \left(\frac{Nh^3}{4\pi V}\right)^{1/3}$$

$$U = \frac{6\pi V}{h^3 m}\frac{p_F^5}{5} = \frac{6\pi V}{5h^3 m}\left(\frac{Nh^3}{4\pi V}\right)^{5/3}$$

$$= \frac{3Nh^2}{10m}\left(\frac{N}{4\pi V}\right)^{2/3}$$