Physics 5500 2nd Midterm Exam 2013 NAME ______________________

Problem 1.
A spin-$\frac{1}{2}$ particle with spin in the direction specified by the spherical angles θ and ϕ can be represented by the normalized spinor

$$\chi(\theta,\phi) = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{+i\phi/2} \end{pmatrix}.$$

Some useful properties of this spinor and of the spin matrices for spin $\frac{1}{2}$ are given on the last page of this exam.

(A) Suppose the spin component S_z of the particle with spinor $\chi(\theta,\phi)$ is measured. What are the possible results of the measurement? For each possible result, what is the probability of obtaining that result and what is the normalized spinor after the measurement?

$$\chi(\theta,\phi) = \cos(\theta/2)\, e^{-i\phi/2}\begin{pmatrix}1\\0\end{pmatrix} + \sin(\theta/2)\, e^{+i\phi/2}\begin{pmatrix}0\\1\end{pmatrix}$$

$S_z = +\frac{1}{2}$ probability: $|\cos(\theta/2)e^{-i\phi/2}|^2 = \cos^2(\theta/2)$

spinor: $\begin{pmatrix}1\\0\end{pmatrix}$

$S_z = -\frac{1}{2}$ probability: $|\sin(\theta/2)\, e^{-i\phi/2}|^2 = \sin^2\theta/2$

spinor: $\begin{pmatrix}0\\1\end{pmatrix}$

(B) Verify explicitly that $\binom{1/\sqrt{2}}{i/\sqrt{2}}$ is an eigenvector of S_y and determine its eigenvalue.

$$S_y \begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} = \frac{1}{2}\hbar \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix}$$

$\Rightarrow$ eigenvalue is $+\frac{1}{2}\hbar$

(C) Suppose the spin component S_y of the particle with spinor $\chi(\theta,\phi)$ is measured. What are the possible results of the measurement? For each possible result, what is the probability of obtaining that result and what is the normalized spinor after the measurement?

$$\chi(\theta,\phi) = \frac{\cos(\theta/2)e^{-i\phi/2} - i\sin(\theta/2)\, e^{+i\phi/2}}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix} + \frac{\cos(\theta/2)e^{-i\phi/2} + i\sin(\theta/2)e^{i\phi/2}}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix}$$

$S_y = +\frac{1}{2}\hbar$ probability: $\dfrac{|\cos(\theta/2)\, e^{-i\phi/2} - i\sin(\theta/2)e^{+i\phi/2}|^2}{2}$

spinor: $\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix}$

$S_y = -\frac{1}{2}\hbar$ probability: $\dfrac{|\cos(\theta/2)\, e^{-i\phi/2} + i\sin(\theta/2)\, e^{+i\phi/2}|^2}{2}$

spinor: $\begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix}$

(D) Calculate the expectation value $\langle S_z \rangle_\chi$ of the operator S_z in the spinor $\chi(\theta, \phi)$.

$$\langle S_z \rangle_\chi = \chi(\theta,\phi)^\dagger S_z \chi(\theta,\phi)$$

$$= \left(\cos(\theta/2) e^{-i\phi/2} \quad \sin(\theta/2) e^{+i\phi/2}\right)^* \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{+i\phi/2} \end{pmatrix}$$

$$= \frac{1}{2}\hbar \left(\cos(\theta/2) e^{+i\phi/2} \quad \sin(\theta/2) e^{-i\phi/2}\right) \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{+i\phi/2} \end{pmatrix}$$

$$= \frac{1}{2}\hbar \left(\cos^2(\theta/2) - \sin^2(\theta/2)\right)$$

(E) What measurement or set of measurements will give a result that is approximately equal to the expectation value in part (D)?

take many particles with the same spin state $\chi(\theta,\phi)$

measure S_z for each of them,

which will give either $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$

average the measurements

(F) In Dirac notation, the expectation value of S_z in a normalized spinor χ can be expressed as

$$\langle S_z \rangle_\chi = \langle \chi | S_z | \chi \rangle.$$

Prove that because S_z is a hermitian operator, its expectation value in any normalized state $|\chi\rangle$ must be real.

$$\langle S_z \rangle_\chi^* = \langle \chi | S_z | \chi \rangle^*$$

$$= (|\chi\rangle)^\dagger S_z^\dagger (\langle \chi |)^\dagger$$

$$= \langle \chi | S_z | \chi \rangle \quad \text{because } S_z \text{ is hermitian } S_z^\dagger = S_z$$

$$= \langle S_z \rangle_\chi$$

therefore $\langle S_z \rangle_\chi$ is real

The magnetic moment of the spin-$\frac{1}{2}$ particle is $\vec{\mu} = (ge/2m)\vec{S}$. Its wavepacket is localized near $z = 0$ and moving along the x axis. Its initial spinor $\chi(\theta, \phi)$ is given on page 1. The particle passes through a Stern-Gerlach magnet that shifts its trajectory vertically by an amount z proportional to the z compnent of its magnetic moment: $z = C\mu_z$, where C is a constant.

(G) After the particle passes through the magnet, its z coordinate is measured by allowing it to hit a collecting plate. If the vertical position is observed to be z_0, what measured values of μ_z and S_z can be inferred from the measurement of z?

$$z = C\mu_z, \quad \mu_z = (ge/2m) S_z$$

measurement: $z = z_0 \implies \mu_z = \frac{1}{C} z_0$

$$S_z = \frac{1}{C(ge/2m)} z_0$$

(H) Suppose 1000 particles with initial spinor $\chi(\theta, \phi)$ pass through the Stern-Gerlach magnet and strike the collecting plate. What are all the values of S_z that will be observed? For each value of S_z, what is the expected number of particles with that value?

$S_z = \frac{1}{2}\hbar$: probability: $|\cos(\theta/2)\, e^{-i\phi/2}|^2 = \cos^2(\theta/2)$

expected number: $1000 \cos^2(\theta/2)$

$S_z = -\frac{1}{2}\hbar$ probability: $|\sin(\theta/2)\, e^{+i\phi/2}|^2 = \sin^2(\theta/2)$

expected number: $1000 \sin^2(\theta/2)$

The collecting plate is now replaced by an absorber that blocks a particle whose trajectory is shifted downward (because S_z is negative) but allows a particle whose trajectory is shifted upward to be transmitted.

(I) If the initial spinor of the particle entering the Stern-Gerlach magnet is $\chi(\theta, \phi)$, what is the probability that it will be transmitted past the absorber? If it is transmitted, what will the normalized spinor for the transmitted particle be?

probability: $|\cos(\theta/2)\, e^{-i\phi/2}|^2 = \cos^2(\theta/2)$

spinor: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Problem 2.
The Schroedinger equation in spherical coordinates for an electron with energy E_n that is bound in a hydrogen atom can be expressed in the form

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 r + \frac{L_x^2+L_y^2+L_z^2}{2mr^2} - \frac{K}{r}\right]\psi = E_n\psi,$$

where $K = e^2/(4\pi\epsilon_0)$ and L_x, L_y, and L_z are angular differential operators.

(A) The spherical harmonic $Y_{\ell m}(\theta,\phi)$ has subscripts ℓ and m that specify the eigenvalues of the differential operators $L_x^2+L_y^2+L_z^2$ and L_z. What are their eigenvalues? What are the possible values of the quantum numbers ℓ and m?

$L_x^2+L_y^2+L_z^2$ has eigenvalue $\ell(\ell+1)\hbar^2$, $\ell = 0, 1, 2, \ldots$

L_z " $m\hbar$, $m = -\ell, -\ell+1, \ldots, +\ell$

(B) Suppose the wave function has the form $\psi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi)$, where $Y_{\ell m}$ is a spherical harmonic. Reduce the Schroedinger equation above to a differential equation for $R(r)$.

replace $L_x^2+L_y^2+L_z^2$ by its eigenvalue

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 r + \frac{\ell(\ell+1)\hbar^2}{2mr^2} - \frac{K}{r}\right] R(r)\, Y_{\ell m}(\theta,\phi) = E_n R(r)\, Y_{\ell m}(\theta,\phi)$$

$$\left[-\frac{\hbar^2}{2}\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 r + \frac{\ell(\ell+1)\hbar^2}{2mr^2} - \frac{K}{r}\right] R(r) = E_n R(r)$$

For some choices of the quantum numbers ℓ and m, the radial Schroedinger equation reduces to

$$\left[-\frac{\hbar^2}{2m}\left(\frac{d}{dr}\right)^2 - \frac{K}{r}\right] rR = E_n rR.$$

(C) At large r, there are two approximate solutions for $rR(r)$ that can be expressed in the form $\exp(-r/a)$ and $\exp(+r/a)$. Deduce the value of the positive variable a. Why is the approximate solution $\exp(+r/a)$ unphysical?

$$\left[-\frac{\hbar^2}{2m}\left(\frac{d}{dr}\right)^2 - \frac{K}{r}\right] e^{\pm r/a} = \left[-\frac{\hbar^2}{2m}\left(\pm\frac{1}{a}\right)^2 - \frac{K}{r}\right] e^{\pm r/a}$$

$$\approx -\frac{\hbar^2}{2ma^2}\, e^{\pm r/a} \quad \text{at large } r$$

$$\Rightarrow \quad E_n = -\frac{\hbar^2}{2ma^2} \quad \Rightarrow \quad a = \frac{\hbar}{\sqrt{2m(-E_n)}}$$

A solution that behaves like $e^{+r/a}$ at larger r is not normalizable.

An exact solution to the simple differential equation above part (C) can be expressed as the approximate solution multiplied by a power series in r:

$$rR(r) = \left(\sum_{j=0}^{\infty} c_j r^j\right) \exp(-r/a).$$

The differential equation can then be reduced to

$$\sum_{j=2}^{\infty} \left[j(j+1)c_{j+1} - \frac{2}{a} j c_j + \frac{2mK}{\hbar^2} c_j \right] r^j = 0.$$

(D) Deduce a recursion relation for the coefficients c_j.

$$j(j+1)C_{j+1} - \frac{2}{a} j C_j + \frac{2mK}{\hbar^2} C_j$$

$$C_{j+1} = \frac{\frac{2}{a} j - \frac{2mK}{\hbar^2}}{j(j+1)} C_j$$

(E) Verify that an approximate solution to the recursion relation for large j is

$$c_j \approx \frac{1}{j!}\left(\frac{2}{a}\right)^j.$$

$$\frac{\frac{2}{a} j - \frac{2mK}{\hbar^2}}{j(j+1)} C_j = \frac{\frac{2}{a} j - \frac{2mK}{\hbar^2}}{j(j+1)} C_j \approx \frac{2}{a(j+1)} C_j$$

$$= \frac{2}{a(j+1)} \frac{1}{j!}\left(\frac{2}{a}\right)^j = \frac{1}{(j+1)!}\left(\frac{2}{a}\right)^{j+1} = C_{j+1}$$

(F) The corresponding approximate solution to the differential equation at large r is

$$rR(r) \approx \left[\sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{2r}{a}\right)^j\right] \exp(-r/a).$$

Why is such a solution unphysical?

$$\left[\sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{2r}{a}\right)^j\right] e^{-r/a} = e^{2r/a} \cdot e^{-r/a} = e^{+r/a}$$

Such a solution is not normalizable

(G) Use the recursion relation obtained in part (D) to deduce the quantization condition that determines the energy eigenvalues E_n. (The final answer $E_n = -\mathrm{Ry}/n^2$ can be obtained by combining this result with that of part (C), but you do NOT need to do this.)

The power series must terminate at some value $j = n$.

So $C_{n+1} = 0$ but $C_n \neq 0$.

The recursion relation implies $0 = \dfrac{\frac{2}{a} n - \frac{2mK}{\hbar^2}}{n(n+1)} C_n$

Since $C_n \neq 0$, we must have $\frac{2}{a} n - \frac{2mK}{\hbar^2} = 0$

$$a = n \frac{\hbar^2}{mK}$$

Problem 3.

The components J_x, J_y, and J_z of an angular momentum vector $\vec{J}$ are hermitian operators that satisfy the angular momentum algebra. The operator $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ commutes with each component of $\vec{J}$.

(A) The raising and lowering operators for J_z are $J_\pm = J_x \pm iJ_y$. Use the angular momentum algebra to derive the commutation relation

$$[J_z, J_-] = -\hbar J_-.$$

$$[J_z, J_-] = [J_z, J_x - iJ_y] = [J_z, J_x] - i[J_z, J_y]$$
$$= i\hbar J_y - i(-i\hbar J_x) = -\hbar(J_x - iJ_y) = -\hbar J_-$$

(B) Suppose $|\psi\rangle$ is an eigenstate of J_z with eigenvalue $m\hbar$. Use the commutator in part (A) to derive a simpler expression for $J_z J_-|\psi\rangle$. What are its implications for the state $J_-|\psi\rangle$?

$J_z J_- - J_- J_z = -\hbar J_- \qquad J_z J_- = J_- J_z - \hbar J_-$

$J_z J_- |\psi\rangle = J_- J_z |\psi\rangle - \hbar J_- |\psi\rangle = J_-(m\hbar|\psi\rangle) - \hbar J_-|\psi\rangle = (m-1)\hbar J_-|\psi\rangle$

$\Longrightarrow$ either $J_-|\psi\rangle = 0$

or $J_-|\psi\rangle$ is an eigenstate of J_z with eigenvalue $(m-1)\hbar$

(C) Use the angular momentum algebra to derive the equation

$$J_+J_- = \vec{J}^2 - J_z^2 + \hbar J_z.$$

$J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y] = J_x^2 + J_y^2 - i \cdot i\hbar J_z$

$\vec{J}^2 - J_z^2 + \hbar J_z = (J_x^2 + J_y^2 + \cancel{J_z^2}) - \cancel{J_z^2} + \hbar J_z = J_x^2 + J_y^2 + \hbar J_z$

(D) Let $|j, m\rangle$ be a simultaneous eigenstate of $\vec{J}^2$ and J_z with eigenvalues $j(j+1)\hbar^2$ and $m\hbar$, respectively. The state $|j, m\rangle$ is normalized:

$$\langle j, m|j, m\rangle = (|j, m\rangle)^\dagger |j, m\rangle = 1.$$

Use the identity in part (C) to determine the normalization of the state $J_-|j, m\rangle$.

$\| J_-|jm\rangle \|^2 = (J_-|jm\rangle)^\dagger J_-|jm\rangle = \langle jm| J_+ J_- |jm\rangle$

$= \langle jm| \vec{J}^2 - J_z^2 + \hbar J_z |jm\rangle$

$= \langle jm| \left(j(j+1)\hbar^2 - (m\hbar)^2 + \hbar \cdot m\hbar\right) |jm\rangle$

$= [j(j+1) - m^2 + m]\hbar^2 \langle jm|jm\rangle = [j(j+1) - m^2 + m]\hbar^2$

$\Longrightarrow$ norm of $J_-|jm\rangle$ is $\sqrt{j(j+1) - m^2 + m}\,\hbar$

The Hamiltonian for the 1-dimensional harmonic oscillator can be expressed in the form

$$\hat{H} = \hbar\omega\left(\hat{A}^\dagger\hat{A} + \frac{1}{2}\right).$$
$$\hat{A} = \frac{1}{\sqrt{2m\hbar\omega}}\left(\hat{P} - im\omega\hat{X}\right).$$

(E) Use the commutation relation $[\hat{X}, \hat{P}] = i\hbar$ to derive the commutation relation $[\hat{A}, \hat{A}^\dagger] = 1$.

$$[A, A^\dagger] = \left(\frac{1}{\sqrt{2m\hbar\omega}}\right)^2 [P - im\omega X, P + im\omega X]$$
$$= \frac{1}{2m\hbar\omega}\left(im\omega[P, X] - im\omega[X, P]\right)$$
$$= \frac{1}{2m\hbar\omega}\left(im\omega(-i\hbar) - im\omega \cdot i\hbar\right) = 1$$

(F) Use the commutation relation $[\hat{A}, \hat{A}^\dagger] = 1$ to show that $\hat{A}$ is a lowering operator for the Hamiltonian:

$$[\hat{H}, \hat{A}] = -\hbar\omega\hat{A}.$$

$$[H, A] = [\hbar\omega(A^\dagger A + \tfrac{1}{2}), A] = \hbar\omega[A^\dagger A, A]$$
$$= \hbar\omega\left(A^\dagger A \cdot A - A \cdot A^\dagger A\right) = \hbar\omega[A^\dagger, A]A$$
$$= \hbar\omega(-1)A = -\hbar\omega A$$

(G) Suppose $|n\rangle$ is an eigenvector of the Hamiltonian with eigenvalue E_n. Use the commutator in part (F) to derive a simpler expression for $\hat{H}\hat{A}|n\rangle$. What are its implications for the state $\hat{A}|n\rangle$?

$$HA - AH = -\hbar\omega A \qquad HA = AH - \hbar\omega A$$

$$HA|n\rangle = AH|n\rangle - \hbar\omega A|n\rangle = A(E_n|n\rangle) - \hbar\omega A|n\rangle = (E_n - \hbar\omega)A|n\rangle$$

$\Longrightarrow$ either $A|n\rangle = 0$

or $A|n\rangle$ is an eigenvector of H with eigenvalue $E_n - \hbar\omega$

(H) Suppose the state $|\psi\rangle$ satisfies $\hat{A}|\psi\rangle = 0$. Express this condition as a differential equation for the wavefunction $\psi(x)$ associated with the state $|\psi\rangle$.

$$\hat{A}|\psi\rangle = 0 \quad \Longrightarrow \quad (\hat{P} - im\omega\hat{X})|\psi\rangle = 0$$

$$\Longrightarrow \left(-i\hbar\frac{\partial}{\partial x} - im\omega \cdot x\right)\psi(x) = 0$$

$$\frac{\partial}{\partial x}\psi(x) = -\frac{m\omega x}{\hbar}\psi(x)$$

Spin matrices and spinors for spin $\frac{1}{2}$

Spin matrices:

$$S_x = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \tfrac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Commutation relations:

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y.$$

Square of the spin vector:

$$\vec{S}^2 \equiv S_x^2 + S_y^2 + S_z^2 = \tfrac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Raising and lowering operators for S_z: $S_\pm = S_x \pm iS_y$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Normalized eigenvectors of S_x:

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

Normalized eigenvectors of S_y:

$$\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}.$$

Normalized eigenvectors of S_z:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Spinor with spin in the direction (θ, ϕ):

$$\begin{aligned}
\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{+i\phi/2} \end{pmatrix}
&= \frac{\cos(\theta/2)e^{-i\phi/2} + \sin(\theta/2)e^{+i\phi/2}}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\
&\quad + \frac{\cos(\theta/2)e^{-i\phi/2} - \sin(\theta/2)e^{+i\phi/2}}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}. \\
&= \frac{\cos(\theta/2)e^{-i\phi/2} - i\sin(\theta/2)e^{+i\phi/2}}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \\
&\quad + \frac{\cos(\theta/2)e^{-i\phi/2} + i\sin(\theta/2)e^{+i\phi/2}}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \\
&= \cos(\theta/2)e^{-i\phi/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\theta/2)e^{+i\phi/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$