Physics 5500 1st Midterm Exam 2013 NAME ____________________

33 **Problem 1.**

The classical model for the hydrogen atom is an electron orbiting the proton like a planet around the sun. The energy E of a circular orbit with radius r and velocity v is

$$E = \frac{1}{2} m_e v^2 - \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r} = -\frac{1}{2} m_e v^2 = -\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r}.$$

Newton's equations have been used to eliminate r or v in the last two expressions.

5 (A) Use the expressions for the energy to express the velocity v of the circular orbit in terms of its radius r. What is the period T of a circular orbit with radius r?

$$-\frac{1}{2} m_e v^2 = -\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r} \Rightarrow v^2 = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{m_e r}, \quad v = \sqrt{\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{m_e r}}$$

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{m_e r}{e^2/4\pi\epsilon_0}}$$

3 (B) According to Maxwell's Equations, what is the frequency of electromagnetic radiation emitted by an electron in a circular orbit with radius r?

The frequency is the orbital frequency of the electron

$$\nu = \frac{1}{T} = \frac{v}{2\pi r}$$

Suppose the energy of the electron in a circular orbit can only have the discrete values $E_n = -\mathrm{Ry}/n^2$, where n is a positive integer.

4 (C) A photon is emitted when an electron makes the transition from the $n = 5$ level to the $n = 3$ level. Express the frequency of the photon in terms of Ry.

energy conservation: $E_5 = E_3 + h\nu$

$$\nu = \frac{E_5 - E_3}{h} = \frac{1}{h}\left[\left(-\frac{Ry}{25}\right) - \left(-\frac{Ry}{9}\right)\right] = \left(\frac{1}{9} - \frac{1}{25}\right)\frac{Ry}{h}$$

4 (D) Use the assumption that $E_n = -\mathrm{Ry}/n^2$ to express the discrete values of the radius r, the velocity v, and the angular momentum $L = m_e v r$ in terms of Ry.

$$-\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r_n} = -\frac{Ry}{n^2} \Rightarrow r_n = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{n^2}{2Ry}$$

$$-\frac{1}{2} m_e v^2 = -\frac{Ry}{n^2} \Rightarrow v_n^2 = \frac{2Ry}{m_e n^2}, \quad v = \sqrt{\frac{2Ry}{m_e}}\frac{1}{n}$$

$$L_n = m_e v_n r_n = m_e\sqrt{\frac{2Ry}{m_e}}\frac{1}{n}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{n^2}{2Ry} = \left(\frac{e^2}{4\pi\epsilon_0}\right)\sqrt{\frac{2Ry}{m_e}}\,n$$

Bohr's expression for the Rydberg constant Ry in terms of fundamental constants is

$$\mathrm{Ry} = \frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2.$$

3 (E) This expression can be deduced by assuming that the angular momentum L_n has particularly simple discrete values. What are those discrete values?

$n\hbar$, where n is a positive integer

3 (F) Bohr's expression for the Rydberg constant can also be deduced by setting the classical frequency of electromagnetic radiation in part (B) equal to the frequency of the photon emitted in some transition between energy levels of the electron. What is the transition?

transition from $n = N+1$ to $n = N$
in the limit $N \to \infty$

4 (G) The electron makes a transition from the $N+2$ level to the N level, where N is a very large integer. Express the change in the electron energy in a simple form proportional to a power of N. (The binomial expansion is $(1+\epsilon)^p = 1 + p\epsilon + \ldots$ for small ϵ.)

$$E_N - E_{N+2} = \left(-\frac{Ry}{N^2}\right) - \left(-\frac{Ry}{(N+2)^2}\right) = Ry\left[\frac{1}{(N+2)^2} - \frac{1}{N^2}\right]$$

$$= \frac{Ry}{N^2}\left[\frac{N^2}{(N+2)^2} - 1\right] = \frac{Ry}{N^2}\left[\left(1+\frac{2}{N}\right)^{-2} - 1\right]$$

$$= \frac{Ry}{N^2}\left[\left(1 - 2\cdot\frac{2}{N} + \cdots\right) - 1\right] \approx -\frac{4}{N^3}Ry$$

3 (H) Balmer's empirical formula for the wavelengths λ_k in the visible spectrum of the hydrogen atom was

$$\frac{1}{\lambda_k} = \frac{1}{\lambda_B}\left(\frac{1}{4} - \frac{1}{(k+2)^2}\right), \qquad k = 1, 2, \ldots.$$

What transitions between electron energy levels must be responsible for the visible wavelengths?

transition from $n = k+2$ to $n = 2$

4 (I) What is the energy of a photon with wavelength λ_k? Use this to express Balmer's constant λ_B in part (H) in terms of Ry.

photon energy: $E_k = \frac{hc}{\lambda_k}$

$$= \frac{hc}{\lambda_B}\left(\frac{1}{4} - \frac{1}{(k+2)^2}\right)$$

decrease in electron energy: $\left(-\frac{Ry}{(k+2)^2}\right) - \left(-\frac{Ry}{2^2}\right) = Ry\left(\frac{1}{4} - \frac{1}{(k+2)^2}\right)$

conservation of energy $\Rightarrow \frac{hc}{\lambda_B} = Ry \Rightarrow \lambda_B = \frac{hc}{Ry}$

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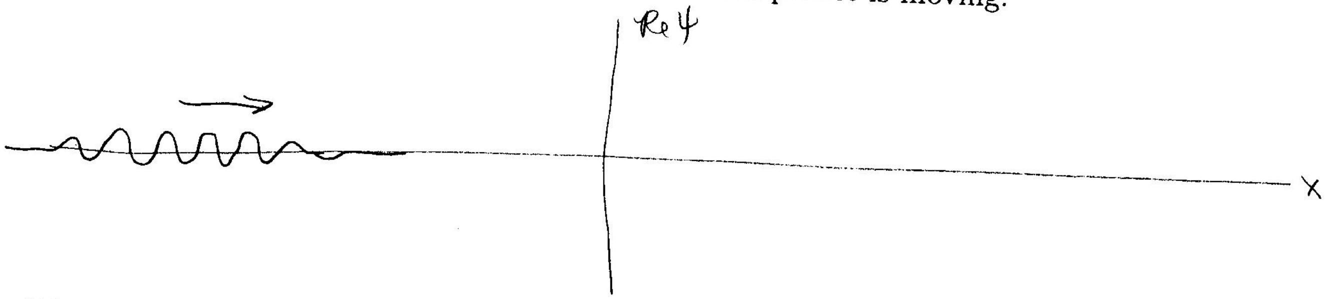
Problem 2.

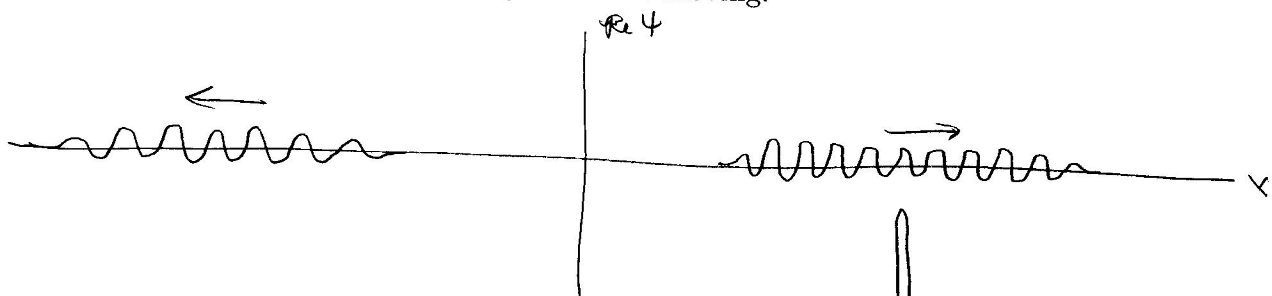
A particle moving to the right with positive energy E scatters from a step potential $V(x)$ that changes from 0 for $x < 0$ to a lower energy $-V_0$ for $x > 0$.

The scattering probabilities can be determined from a solution $\Psi(x,t)$ to the Schroedinger equation that is normalized: $\int_{-\infty}^{+\infty} dx\, |\Psi(x,t)|^2 = 1$.

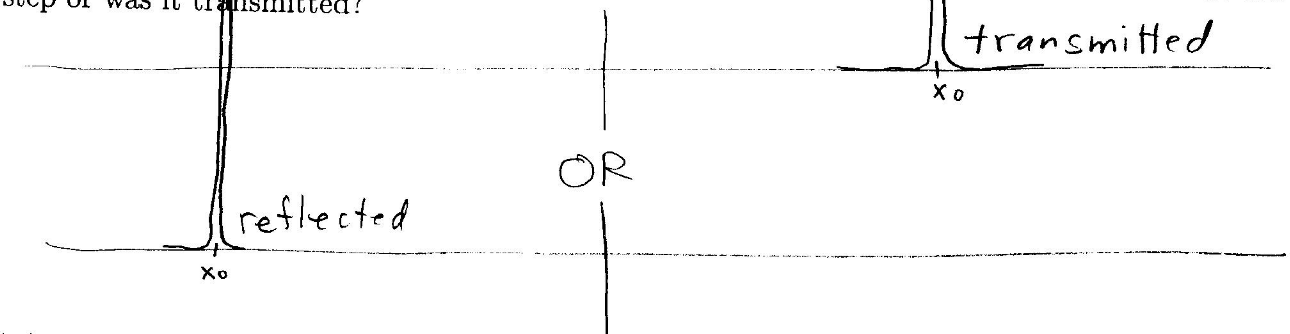
3 (A) At an initial time $t = 0$, when the particle is still approaching the step, the wavefunction $\Psi(x,0)$ is a wave packet. Sketch $\mathrm{Re}\Psi(x,0)$ as a function of x, indicating the position of the step. Draw an arrow indicating the direction the wave packet is moving.



3 (B) At a later time $t = T$, after the particle has scattered from the step, the wavefunction $\Psi(x,T)$ is the sum of two wave packets. Sketch $\mathrm{Re}\Psi(x,T)$ as a function of x. Draw arrows indicating the directions the two wave packets are moving.



3 (C) Suppose the position of the particle is measured at time T when the wavefunction has the form sketched in part (B). Indicate a possible result x_0 of the measurement on the x axis. Sketch the wavefunction immediately after the measurement. Was the particle reflected at the step or was it transmitted?



3 (D) The wavefunction at time T before any measurement can be expressed as

$$\Psi(x,T) = \psi_{\text{left}}(x) + \psi_{\text{right}}(x),$$

where $\psi_{\text{left}}(x)$ is nonzero only far to the left of the step and $\psi_{\text{right}}(x)$ is nonzero only far to the right of the step. What is the probability R that a measurement of the position at time T reveals that the particle was reflected?

$$R = \int_{-\infty}^{0} dx\, |\psi_{left}(x)|^2$$

The scattering probabilities can also be determined from a solution $\psi(x)$ to the time-independent Schroedinger equation with energy E.

3 (E) The most general solution for $\psi(x)$ in the region $x > 0$ has the form

$$\psi(x) = Ce^{ik_2 x} + De^{-ik_2 x},$$

where C and D are arbitrary coefficients. Derive the expression for k_2?

$$-\frac{\hbar^2}{2m}\psi'' + (-V_0)\psi = E\psi$$

$$\psi = e^{\pm ik_2 x}; \quad -\frac{\hbar^2}{2m}(\pm ik_2)^2 + (-V_0) = E$$

$$\frac{\hbar^2 k^2}{2m} = E + V_0 \qquad \boxed{k_2 = \frac{\sqrt{2m(E+V_0)}}{\hbar}}$$

3 (F) Write down the most general solution for $\psi(x)$ in the region $x < 0$, with the only unknowns being arbitrary coefficients A and B.

$$\boxed{\psi(x) = Ae^{ik_1 x} + Be^{-ik_1 x}}$$

$$-\frac{\hbar^2}{2m}\psi'' + 0\cdot\psi = E\psi$$

$$\psi = e^{\pm ik_1 x}, \quad -\frac{\hbar^2}{2m}(\pm ik_1)^2 + 0 = E$$

$$\frac{\hbar^2 k_1^2}{2m} = E \qquad \boxed{k_1 = \frac{\sqrt{2mE}}{\hbar}}$$

3 (G) What constraints on the coefficients A, B, C, and D are imposed by the requirement that the wavefunction describes the scattering of a particle that approaches the step from the left?

no incoming wave from right

$\Longrightarrow$ no $e^{-ik_2 x}$ term $\Longrightarrow$ $\boxed{D = 0}$

4 (H) What constraints on the coefficients are imposed by the requirement that the solutions match at $x = 0$? Express them as linear equations in A, B, C, and D.

$\psi(x)$ continuous at $x = 0$:

$$\boxed{A + B = C + D}$$

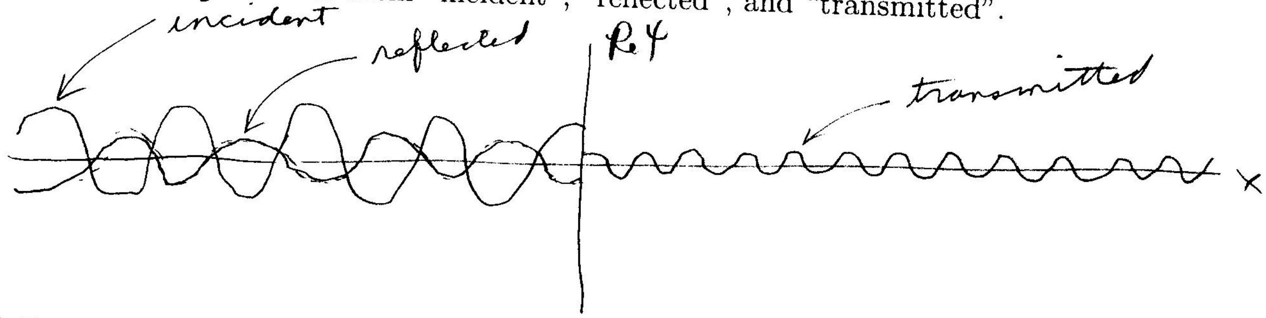
$\psi'(x)$ continuous at $x = 0$:

$$x < 0: \quad \psi'(x) = Aik_1 e^{ik_1 x} + B(-ik_1)e^{-ik_1 x}$$

$$x > 0: \quad \psi'(x) = Cik_2 e^{ik_2 x} + D(-ik_2)e^{-ik_2 x}$$

$$\boxed{ik_1 A - ik_1 B = ik_2 C - ik_2 D}$$

3 (I) Sketch the real parts of the wavefunctions as functions of x for the incident, transmitted, and reflected waves. Label them "incident", "reflected", and "transmitted".



3 (J) For each of the three waves, identify the speed and the direction with which its probability is flowing.

incident: moving to right with speed $v_1 = \frac{\hbar k_1}{m} = \sqrt{\frac{2E}{m}}$

reflected: " left v_1

transmitted: " right $v_2 = \frac{\hbar k_2}{m} = \sqrt{\frac{2(E+V_0)}{m}}$

3 (K) Express the probability T for the particle to be transmitted at the step in terms of A, B, C, D, and other parameters in the wavefunctions in parts (E) and (F).

$$T = \frac{|C|^2 v_2}{|A|^2 v_1} = \frac{|C|^2}{|A|^2}\sqrt{\frac{E+V_0}{E}}$$

33 **Problem 3.**

A particle E is bound in a square-well potential $V(x)$ that is 0 in the regions $-\infty < x < -a$ and $+a < x < +\infty$ and has a negative value $-V_0$ for $-a < x < +a$. The particle has a negative energy E in the range $-V_0 < E < 0$.

The bound state can be described by a solution $\psi(x)$ to the time-independent Schroedinger equation with energy E.

4 (A) The most general solution for $\psi(x)$ in the region $-a < x < +a$ is

$$\psi(x) = Ce^{ikx} + De^{-ikx},$$

where $k = \sqrt{2m(V_0 + E)}/\hbar$ and C and D are arbitrary coefficients. Write down the most general solution $\psi(x)$ in the region $+a < x < +\infty$, with the only unknowns being arbitrary coefficients F and G.

$$\boxed{\psi(x) = Fe^{-\kappa x} + Ge^{+\kappa x}}$$

$$-\frac{\hbar^2}{2m}\psi'' + 0\cdot\psi = E\psi$$

$$\psi = e^{\pm\kappa x}: \quad -\frac{\hbar^2}{2m}(\pm\kappa)^2 + 0 = E$$

$$-\frac{\hbar^2}{2m}\kappa^2 = E \implies \boxed{\kappa = \frac{\sqrt{2m(-E)}}{\hbar}}$$

3 (B) What constraints on the coefficients C, D, F, and G are imposed by the condition that $\psi(x)$ be normalizable?

$e^{\kappa x} \to \infty$ as $x \to +\infty$

$\implies$ no $e^{\kappa x}$ term $\implies$ $\boxed{G = 0}$

4 (C) The equations on all the arbitrary coefficients that are imposed by the condition that the solutions match at $x = +a$ and at $x = -a$ can be reduced to 4 linear equations in 4 unknowns. However the fact that the potential $V(x)$ is an even function of x can be exploited to reduce the problem to 2 equations in 2 unknowns for even wavefunctions and 2 equations in 2 unknowns for odd wavefunctions. The equations for even wavefunctions are

$$e^{-\kappa a}A = 2\cos(ka)C,$$
$$\kappa e^{-\kappa a}A = 2k\sin(ka)C,$$

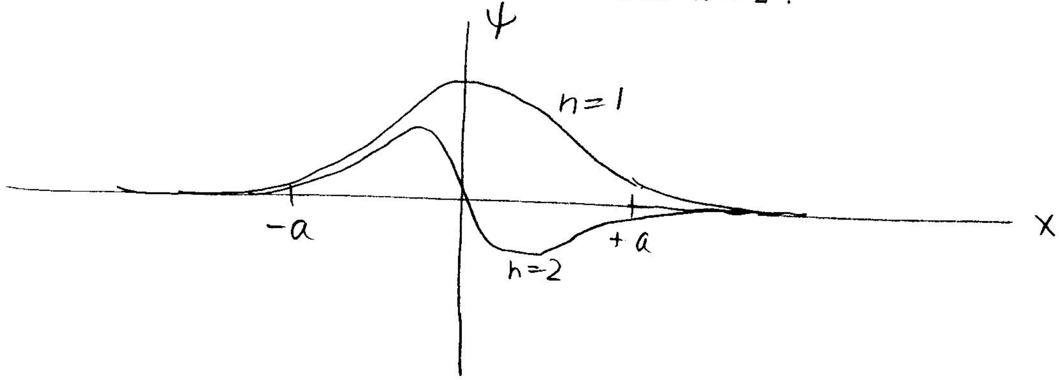
where k and κ are known functions of the energy E. What single equation must be satisfied by the energy E in order for it to be the energy of a bound state?

eliminate A and C:

$$\frac{\kappa e^{-\kappa a}\cancel{A}}{e^{-\kappa a}\cancel{A}} = \frac{2k\sin(ka)\cancel{C}}{2\cos(ka)\cancel{C}}$$

$$\boxed{\kappa = k\frac{\sin(ka)}{\cos(ka)}}$$

4 (D) Assume that there are at least 2 bound-state energy levels. Sketch the wavefunctions $\psi_1(x)$ and $\psi_2(x)$ for the ground state and the first excited state. Mark the positions $-a$ and $+a$ on the x axis and label the wavefunctions "$n = 1$" and "$n = 2$".



3 (E) Suppose the particle emits a photon whose frequency ν is measured. What frequency would inform you that the particle had made a transition from the excited state $n = 2$ to the ground state $n = 1$?

energy conservation: $E_2 = E_1 + h\nu$

$$\nu = \frac{E_2 - E_1}{h}$$

3 (F) If the photon in part (F) is observed at time $t = 0$, the particle will definitely be in the ground state with energy E_1 and wavefunction $\psi_1(x)$. What is its Schroedinger wavefunction $\Psi(x, t)$ at later times t?

$$\Psi(x,t) = \psi_1(x)\, e^{-iE_1 t/\hbar}$$

3 (G) The particle is in the ground state at time $t = 0$. Suppose its energy E is measured at a later time $t = T$. What are the possible values of E and what is the probability for each value?

only possible value is E_1

probability is 100%

3 (H) Suppose you are given a wavefunction $\psi(x)$ for the particle that is NOT normalized. What is the probability distribution $P(x)dx$ for a measurement of its position x?

normalization integral: $N = \int_{-\infty}^{+\infty} dx\, |\psi(x)|^2$

probability distribution: $P(x)dx = \frac{1}{N} |\psi(x)|^2 dx$

3 (I) Suppose the wavefunction $\psi(x)$ for the particle is normalized. Express the expectation value $\langle x \rangle$ of the position as an integral over x.

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \; \psi^*(x) \; x \; \psi(x)$$

$$= \int_{-\infty}^{\infty} dx \; x \, |\psi(x)|^2$$

3 (J) For which of the following 1000 measurements of the position will the average of the measurements be close to $\langle x \rangle$?

1. measurements on the same state whose initial wavefunction is $\psi(x)$ at a sequence of 1000 subsequent times
2. measurements on 1000 different states that all have the same initial wavefunction $\psi(x)$
3. both 1. and 2.