Physics 5500 2nd Midterm Exam 2012 NAME ______________________________

Problem 1.
A spin-$\frac{1}{2}$ particle with spin up in the direction of the spherical angles θ and ϕ has the normalized spin wavefunction

$$|\theta,\phi\rangle = \cos(\theta/2)e^{-i\phi/2}|\uparrow\rangle + \sin(\theta/2)e^{+i\phi/2}|\downarrow\rangle = \begin{pmatrix}\cos(\theta/2)e^{-i\phi/2}\\ \sin(\theta/2)e^{+i\phi/2}\end{pmatrix}.$$

Some useful properties of the spin operators for spin $\frac{1}{2}$ are given on the last page of this exam.

(A) The spin component S_z is measured for the spin state $|\theta,\phi\rangle$. What are the possible values of S_z? For each value, what is the probability of obtaining that value?

values of S_z	probability
$+\frac{1}{2}\hbar$	$\left\|\cos\frac{\theta}{2}e^{-i\phi/2}\right\|^2 = \cos^2\frac{\theta}{2}$
$-\frac{1}{2}\hbar$	$\left\|\sin\frac{\theta}{2}e^{+i\phi/2}\right\|^2 = \sin^2\frac{\theta}{2}$

(B) A measurement of S_y in the spin state $|\theta,\phi\rangle$ gives the value $+\frac{1}{2}\hbar$. What is the spin state after that measurement? Express it both as a 2-component spinor and as a linear combination of $|\uparrow\rangle$ and $|\downarrow\rangle$.

measurement collapses wavefunction
to eigenstate of S_y with eigenvalue $+\frac{1}{2}\hbar$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle$$

(C) Suppose a measurement of S_x in the spin state $|\theta,\phi\rangle$ that gives the value $+\frac{1}{2}\hbar$ is followed by a measurement of S_z. What are the possible values of S_z and what are the probabilities for each value?

measurement collapses wavefunction
to eigenstate of S_x with eigenvalue $+\frac{1}{2}\hbar$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

values of S_z	probability
$+\frac{1}{2}\hbar$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$
$-\frac{1}{2}\hbar$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

(D) If the spin is in a magnetic field pointing along the x axis, the Hamiltonian for the spin state has the form

$$H = \omega S_x,$$

where is ω is a constant. The Schrodinger equation for the spin state is $i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$. Express this as a matrix equation for the 2-component spinor $\binom{\psi_1(t)}{\psi_2(t)}$.

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{\hbar\omega}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$H = \omega \tfrac{1}{2}\hbar S_z = \frac{\hbar\omega}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

(E) The time-derivative of the expectation value of S_y in the normalized spin state $|\psi(t)\rangle$ can be expressed as

$$\frac{d}{dt}\langle\psi|S_y|\psi\rangle = -\frac{i}{\hbar}\langle\psi|[S_y, H]|\psi\rangle.$$

Derive this from the Schroedinger equation for $|\psi\rangle$.

$$\frac{d}{dt}\langle\psi|S_y|\psi\rangle = \left(\frac{d}{dt}\langle\psi|\right) S_y|\psi\rangle + \langle\psi| S_y \left(\frac{d}{dt}|\psi\rangle\right)$$

$$= \left(\frac{i}{\hbar}\langle\psi|H\right) S_y|\psi\rangle + \langle\psi| S_y \left(-\frac{i}{\hbar}H|\psi\rangle\right)$$

$$= \frac{i}{\hbar}\langle\psi|(HS_y - S_yH)|\psi\rangle$$

$$= -\frac{i}{\hbar}\langle\psi|[S_y, H]|\psi\rangle$$

(F) Express the time derivative of $\langle\psi|S_y|\psi\rangle$ in terms of the expectation values of the other spin operators S_x and S_z.

$$\frac{d}{dt}\langle\psi|S_y|\psi\rangle = -\frac{i}{\hbar}\langle\psi|[S_y, \omega S_x]|\psi\rangle$$

$$= -\frac{i\omega}{\hbar}\langle\psi|[S_y, S_x]|\psi\rangle$$

$$= -\frac{i\omega}{\hbar}\langle\psi|(-i\hbar S_z)|\psi\rangle$$

$$= -\omega\langle\psi|S_z|\psi\rangle$$

Problem 2.

The time-independent Schroedinger equation for the 3-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2m}\left(\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2+\left(\frac{\partial}{\partial z}\right)^2\right)\psi+\frac{1}{2}m\omega^2(x^2+y^2+z^2)\psi=E\psi.$$

The solution to the Schroedinger equation for the 1-dimensional harmonic oscillator is given on the last page of this exam.

(A) For solutions of the form $\psi(x,y,z)=f(x)g(y)h(z)$, the Schroedinger equation can be expressed as

$$-\frac{\hbar^2}{2m}\left(\frac{f''(x)}{f(x)}+\frac{g''(y)}{g(y)}+\frac{h''(z)}{h(z)}\right)+\frac{1}{2}m\omega^2(x^2+y^2+z^2)=E.$$

Show how this can be reduced to 3 separate differential equations in the variables x, y, and z.

$$\left[-\frac{\hbar^2}{2m}\frac{f''(x)}{f(x)}+\frac{1}{2}m\omega^2x^2\right]+\left[-\frac{\hbar^2}{2m}\frac{g''(y)}{g(y)}+\frac{1}{2}m\omega^2y^2\right]+\left[-\frac{\hbar^2}{2m}\frac{h''(z)}{h(z)}+\frac{1}{2}m\omega^2z^2\right]=E$$

The sum of functions of x only, y only, and z only can be equal to the constant E only if each is separately constant

$$-\frac{\hbar^2}{2m}\frac{f''(x)}{f(x)}+\frac{1}{2}m\omega^2x^2=E_x \qquad -\frac{\hbar^2}{2m}\frac{h''(z)}{h(z)}+\frac{1}{2}m\omega^2z^2=E_z$$

$$-\frac{\hbar^2}{2m}\frac{g''(y)}{g(y)}+\frac{1}{2}m\omega^2y^2=E_y \qquad \text{where } E_x+E_y+E_z=E$$

(B) The differential equation in the variable z can be written in the form

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dz}\right)^2h+\frac{1}{2}m\omega^2z^2h=E_3h.$$

What are the possible eigenvalues E_3? (Be sure to specify the possible values of any quantum number you introduce.) What are the eigenfunctions $h(z)$ for each eigenvalue?

eigenvalues: $E_3=(n+\frac{1}{2})\hbar\omega$, $n=0,1,2,\ldots$

eigenfunctions: $H_n(\beta z)e^{-\beta^2z^2/2}$, $\beta=\sqrt{m\omega/\hbar}$

H_n is Hermite polynomial of degree n

(C) If one looks for solutions of the time-independent Schroedinger equation of the form $\psi(r,\theta,\phi)=R(r)Y_{\ell m}(\theta,\phi)$, where $Y_{\ell m}(\theta,\phi)$ is a spherical harmonic, it reduces to the radial Schroedinger equation

$$-\frac{\hbar^2}{2m}\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2rR+\frac{\ell(\ell+1)\hbar^2}{2mr^2}R+\frac{1}{2}m\omega^2r^2R=ER.$$

This equation is particularly simple if $\ell=0$. In this case, what are the possible eigenvalues E? What are the eigenfunctions $R(r)$ for each eigenvalue?

eigenvalues: $E=(n+\frac{1}{2})\hbar\omega$, $n=0,1,2,\ldots$

eigenfunctions: $\frac{1}{r}H_n(\beta r)\,e^{-\beta^2r^2/2}$, $\beta=\sqrt{m\omega/\hbar}$

H_n is Hermite polynomial of degree n

(D) If the Schroedinger equation is solved by separating the Cartesian coordinates x, y, and z, the energy eigenvalues are determined by three quantum numbers n_x, n_y, and n_z, each of which has values $0, 1, 2, \ldots$:

$$E_{n_x,n_y,n_z} = (n_x + n_y + n_z + \tfrac{3}{2})\hbar\omega.$$

The energy eigenstates $|n_x, n_y, n_z\rangle$ can be labelled by those three quantum numbers. What are the energies for the first three energy levels? (The ground state is the first energy level.) List all the energy eigenstates for the third energy level.

energy levels: $\frac{3}{2}\hbar\omega$, $\frac{5}{2}\hbar\omega$, $\frac{7}{2}\hbar\omega$, ...

3rd energy level: $|2,0,0\rangle$, $|0,2,0\rangle$, $|0,0,2\rangle$,
$|1,1,0\rangle$, $|1,0,1\rangle$, $|0,1,1\rangle$

(E) If the Schroedinger equation is solved by separating the spherical coordinates r, θ, and ϕ, the energy eigenvalues are determined by the angular momentum quantum number ℓ and a radial quantum number n whose values are $0, 1, 2, \ldots$:

$$E_{n,\ell} = (2n + \ell + \tfrac{3}{2})\hbar\omega,$$

The energy eigenstates $|n, \ell, m\rangle$ can be labelled by n, ℓ, and the other angular momentum quantum number m. List all the energy eigenstates for the third energy level $\frac{7}{2}\hbar\omega$.

3rd energy level: $|1,0,0\rangle$

$|0,2,-2\rangle$, $|0,2,-1\rangle$, $|0,2,0\rangle$, $|0,2,-1\rangle$, $|0,2,-2\rangle$

(F) The differential operator

$$A = \frac{1}{\sqrt{2m}}\left(m\omega x + \hbar\frac{\partial}{\partial x}\right)$$

satisfies the commutation relation $[H, A] = -\hbar\omega A$, where H is the Hamiltonian. Suppose $|\psi\rangle$ is an eigenstate of H with eigenvalue E. Use the commutation relation to show that $A|\psi\rangle$ satisfies

$$H\left(A|\psi\rangle\right) = (E - \hbar\omega)\left(A|\psi\rangle\right).$$

$HA = AH - \hbar\omega A$

$HA|\psi\rangle = A\underbrace{H|\psi\rangle}_{E|\psi\rangle} - \hbar\omega A|\psi\rangle$

$HA|\psi\rangle = EA|\psi\rangle - \hbar\omega A|\psi\rangle \qquad H(A|\psi\rangle) = (E - \hbar\omega)A|\psi\rangle$

(G) What can you conclude about the state $A|\psi\rangle$ from the equation derived in part (F)?

either $A|\psi\rangle = 0$

or $A|\psi\rangle$ is an eigenstate of H with eigenvalue $E - \hbar\omega$

Problem 3.

A neutral atom has a single valence electron that is bound in a quantum state with orbital angular momentum quantum number $\ell = 1$.

(A) The orbital angular momentum operator for the $\ell = 1$ electron is $\vec{L}$. What are the possible eigenvalues of $\vec{L}^2$ and L_z?

eigenvalue of $\vec{L}^2$: $\ell(\ell+1)\hbar^2 = 2\hbar^2$

eigenvalues of L_z: $m_\ell \hbar$, $m_\ell = -1, 0, +1$

(B) The spin operator for the electron is $\vec{S}$. What is the value of the spin quantum number s for the electron? What are the possible eigenvalues of $\vec{S}^2$ and S_z?

spin quantum number: $s = \frac{1}{2}$

eigenvalue of $\vec{S}^2$: $s(s+\frac{1}{2})\hbar^2 = \frac{3}{4}\hbar^2$

eigenvalues of S_z: $m_s \hbar$, $m_s = +\frac{1}{2}, -\frac{1}{2}$

(C) The magnetic moment vector for the neutral atom is that of its $\ell = 1$ valence electron:

$$\vec{\mu} = -\frac{e}{2m_e}\left(\vec{L} + 2\vec{S}\right).$$

(I have set $g = 2$, although it is actually 2.002.) What are the possible eigenvalues of μ_z for this atom?

$$\mu_z = -\frac{e}{2m_e}(L_z + 2S_z)$$

eigenvalues: $-\frac{e}{2m_e}(m_\ell \hbar + 2m_s \hbar) = -\frac{e\hbar}{2m_e}(m_\ell + 2m_s)$

$m_\ell = -1, 0, +1$, $m_s = -\frac{1}{2}, +\frac{1}{2}$

(D) A Stern-Gerlach apparatus oriented along the z axis splits a beam of neutral atoms into parallel beams that are displaced in z by a distance proportional to μ_z. Suppose the beam consists of the atoms with an $\ell = 1$ valence electron considered above. How many parallel beams emerge from the Stern-Gerlach apparatus and what are their values of μ_z?

distinct eigenvalue of μ_z: $-\frac{e\hbar}{2m_e} m$, $m = -2, -1, 0, +1, +2$

$\Longrightarrow$ 5 parallel beams

(E) The total angular momentum operator for the $\ell = 1$ valence electron is $\vec{J} = \vec{L} + \vec{S}$. What are the possible eigenvalues of $\vec{J}^2$ and J_z?

$\vec{J}^2 : j(j+1)\hbar^2, \quad j = \frac{1}{2}, \frac{3}{2} \longrightarrow \frac{3}{4}\hbar^2, \ \frac{15}{4}\hbar^2$

$J_z : m_j \hbar, \quad m_j = -j, \dots, +j \Longrightarrow -\frac{1}{2}\hbar, +\frac{1}{2}\hbar \qquad$ if $j = \frac{1}{2}$

$-\frac{3}{2}\hbar, -\frac{1}{2}\hbar, +\frac{1}{2}\hbar, +\frac{3}{2}\hbar \qquad j = \frac{3}{2}$

(F) The raising operator for J_z is $J_+ = J_x + iJ_y$. Use the angular momentum algebra to derive the commutation relation $[J_z, J_+] = \hbar J_+$.

$$[J_z, J_+] = [J_z, J_x + iJ_y] = [J_z, J_x] + i[J_z, J_y]$$
$$= i\hbar J_y + i(-i\hbar J_x) = \hbar(J_x + iJ_y) = \hbar J_+$$

One basis for the angular momentum states of the $\ell = 1$ valence electron consists of the products $|m_\ell\rangle\,|m_s\rangle$ of the normalized eigenstates of J_z and S_z. Another basis consists of the normalized eigenstates $|j, m_j\rangle$ of $\vec{J}^2$ and J_z. The raising operators for the three angular momenta are

$$\begin{aligned}
L_+|m_\ell\rangle &= \sqrt{2 - m_\ell(m_\ell+1)}\,|m_\ell+1\rangle, \\
S_+|m_s\rangle &= \sqrt{\tfrac{3}{4} - m_s(m_s+1)}\,|m_s+1\rangle, \\
J_+|j, m_j\rangle &= \sqrt{j(j+1) - m_j(m_j+1)}\,|j, m_j+1\rangle.
\end{aligned}$$

(G) Given that $|\frac{3}{2}, -\frac{3}{2}\rangle = |-1\rangle\,|-\frac{1}{2}\rangle$, use the raising operators to express $|\frac{3}{2}, -\frac{1}{2}\rangle$ in terms of the basis $|m_\ell\rangle|m_s\rangle$.

$J_+|\frac{3}{2}, -\frac{3}{2}\rangle = \sqrt{\frac{3}{2}\cdot\frac{5}{2} - (-\frac{3}{2})(-\frac{1}{2})}\,|\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{3}\,|\frac{3}{2}, -\frac{1}{2}\rangle$

$L_+|-1\rangle = \sqrt{1\cdot 2 - (-1)\cdot 0}\,|0\rangle = \sqrt{2}\,|0\rangle$

$S_+|-\frac{1}{2}\rangle = \sqrt{\frac{1}{2}\cdot\frac{3}{2} - (-\frac{1}{2})\cdot\frac{1}{2}}\,|+\frac{1}{2}\rangle = |+\frac{1}{2}\rangle$

$J_+|\frac{3}{2}, -\frac{3}{2}\rangle = (L_+|-1\rangle)|-\frac{1}{2}\rangle + |-1\rangle(S_+|-\frac{1}{2}\rangle)$

$\sqrt{3}\,|\frac{3}{2}, -\frac{3}{2}\rangle = (\sqrt{2}\,|0\rangle)|-\frac{1}{2}\rangle + |-1\rangle|+\frac{1}{2}\rangle$

$|\frac{3}{2}, -\frac{3}{2}\rangle = \sqrt{\frac{2}{3}}\,|0\rangle|-\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|-1\rangle|+\frac{1}{2}\rangle$

(H) The only states $|m_\ell\rangle\,|m_s\rangle$ that are eigenstates of $J_z = L_z + S_z$ with eigenvalue $+\frac{1}{2}\hbar$ are $|+1\rangle\,|-\frac{1}{2}\rangle$ and $|0\rangle\,|+\frac{1}{2}\rangle$. Given that

$$|\tfrac{3}{2}, +\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|+1\rangle\,|-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|0\rangle\,|+\tfrac{1}{2}\rangle,$$

deduce the expression for $|\frac{1}{2}, +\frac{1}{2}\rangle$ in the basis $|m_\ell\rangle\,|m_s\rangle$.

$|\frac{1}{2}, +\frac{1}{2}\rangle$ must be a linear combination of $|+1\rangle|-\frac{1}{2}\rangle$ and $|0\rangle|+\frac{1}{2}\rangle$ that is orthogonal to $|\frac{3}{2}, +\frac{1}{2}\rangle$

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|+1\rangle|-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|0\rangle|+\tfrac{1}{2}\rangle$$

Spin operators for spin $\frac{1}{2}$

Spin operators:

$$S_x = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \tfrac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Commutation relations:

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y.$$

Square of the spin vector:

$$\vec{S}^2 \equiv S_x^2 + S_y^2 + S_z^2 = \tfrac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Raising and lowering operators for S_z:

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Normalized eigenvectors of S_x:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Normalized eigenvectors of S_y:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Normalized eigenvectors of S_z:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One-dimensional harmonic oscillator

Schroedinger equation:

$$-\frac{\hbar^2}{2m}\left(\frac{d}{dx}\right)^2 \psi + \frac{1}{2} m\omega^2 x^2 \psi = E\psi.$$

Energy eigenvalues: $E_n = (n + \frac{1}{2})\hbar\omega$,
$n = 0, 1, 2, \ldots$.

Eigenfunctions: $H_n(\beta x)\exp(-\beta^2 x^2/2)$,
$\beta = \sqrt{m\omega/\hbar}$,
$H_n(t)$ is the Hermite polynomial of degree n