Physics 5500 1st Midterm Exam 2012 NAME _____

Problem 1.

Many of the classic experiments that motivated the development of quantum mechanics provided evidence that light consists of particles. Three of the phenomena that were the subjects of these experiments were

- black-body radiation, 2
- Compton effect, 3
- photoelectric effect. 1

(A) For each of these phenomena, which one or more of the following statements is the most appropriate inference from the experiments? Write the appropriate number or numbers after each of the phenomena listed above.

1. Matter absorbs light of frequency ν as if the light consists of particles with energy $h\nu$.
2. Matter emits light of frequency ν as if the light consists of particles with energy $h\nu$.
3. Light of frequency ν scatters from matter as if the light consists of particles with energy $h\nu$ and momentum $h\nu/c$.
4. Matter absorbs light of frequency ν as if the light consists of particles with momentum $h\nu/c$.
5. Matter emits light of frequency ν as if the light consists of particles with momentum $h\nu/c$.

Consider the collision of an X-ray photon with wavelength λ and an electron at rest. Suppose the X-ray photon back-scatters from the electron (its scattering angle is 180°). The back-scattered photon has wavelength λ' and the recoiling electron has momentum p'.

(B) Write down the equation for conservation of energy in the collision. (It should be expressed in terms of λ, λ', p', and physical constants.)

$$\frac{hc}{\lambda} + m_e c^2 = \frac{hc}{\lambda'} + \sqrt{(m_e c^2)^2 + (p'c)^2}$$

(C) Write down the equation for conservation of momentum in the collision.

$$\frac{h}{\lambda} + 0 = -\frac{h}{\lambda'} + p'$$

If ultraviolet light shines on a metal, electrons can be ejected from the metal. The escaped electrons are called *photo-electrons*. The conduction electrons inside the metal have a constant negative potential energy $-U$ and kinetic energies that range from 0 to a maximum E_F that is called the *Fermi energy*. (As an aside, in the case of copper, $U \approx 12$ eV and $E_F \approx 7$ eV.)

(D) Suppose an electron inside the metal with kinetic energy E absorbs a photon of frequency ν. What is its kinetic energy E' while it is still inside the metal? What is its total energy?

kinetic energy: $E' = E + h\nu$

total energy: $E + h\nu - U$

(E) Suppose the energy E' of that electron is large enough that it can escape from the metal. What is the kinetic energy E'' of the photo-electron? What is its maximum possible kinetic energy?

kinetic energy: $E'' = E + h\nu - U$

maximum: $E''_{max} = E_F + h\nu - U$

(F) What is the minimum frequency of light required for it to eject a photo-electron from the metal. (It should be expressed in terms of U, E_F, and physical constants.)

minimum ν corresponds to $E''_{max} = 0$

$$E_F + h\nu_{min} - U = 0$$

$$\nu_{min} = \frac{U - E_F}{h}$$

Problem 2.

A particle is bound in a 1-dimensional square-well potential that is 0 in the region $0 < x < a$ and has a positive value V_0 for $x < 0$ and $x > a$. The energy E of the particle is in the range $0 < E < V_0$.

(A) Write down the most general solution of the Schroedinger equation for energy E in the region $0 < x < a$. (Define any symbols you introduce.)

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

$$-\frac{\hbar^2}{2m}(\pm ik)^2 = E \quad \Rightarrow \quad k = \frac{\sqrt{2mE}}{\hbar}$$

(B) Write down the most general solution of the Schroedinger equation for energy E in the region $x > a$.

$$\psi(x) = C e^{-\kappa x} + D e^{+\kappa x}$$

$$-\frac{\hbar^2}{2m}(\pm\kappa)^2 + V_0 = E \quad \Rightarrow \quad \kappa = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

(C) The probability density cannot diverge as $x \to +\infty$. What constraint does this put on the wavefunction in the region $x > a$?

$|\psi(x)|^2$ cannot diverge as $x \to +\infty$

$$\Rightarrow D = 0$$

(D) Write down (but don't solve) the matching conditions for the wavefunction at $x = a$. (They should be expressed as linear equations for coefficients in the general solutions to the Schroedinger equation.)

$$\psi(a^-) = \psi(a^+): \quad A e^{ika} + B e^{-ika} = C e^{-\kappa a}$$

$$\psi'(a^-) = \psi'(a^+): \quad A(ike^{ika}) + B(-ike^{-ika}) = C(-\kappa e^{-\kappa a})$$

If the walls of the potential well are infinitely high ($V_0 \to \infty$), the energy eigenvalues and the normalized wavefunctions are simple:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},$$
$$\psi_n(x) = \sqrt{2/a}\sin\frac{n\pi x}{a}.$$

(E) Suppose the energy of the particle is measured at time 0 to be $2\pi^2\hbar^2/ma^2$. What is the wavefunction $\psi(x)$ immediately after the measurement? What is the wavefunction $\Psi(x,t)$ at a later time t?

$E_n = \frac{2\pi^2\hbar^2}{ma^2} \Longrightarrow n=2$

$\Psi(x,t=0) = \psi_2(x) = \sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}$

$\Psi(x,t) = \psi_2(x)\, e^{-iE_2t/\hbar} = \sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\, e^{-i\frac{2\pi^2\hbar}{ma^2}t}$

(F) Suppose the particle in the $n = 3$ energy level makes a transition to the $n = 1$ energy level by emitting a photon. What is the frequency of the photon?

photon energy: $E_3 - E_1 = \frac{9\pi^2\hbar^2}{2ma^2} - \frac{\pi^2\hbar^2}{2ma^2} = \frac{4\pi^2\hbar^2}{ma^2}$

frequency: $\nu = \frac{E_3-E_1}{h} = \frac{4\pi^2\hbar^2}{ma^2 h} = \frac{h}{ma^2}$

(G) Suppose the particle is in the $n = 5$ energy level and its position x is measured. What are the possible values of x and what is their probability distribution? Express the condition that the total probability is 1 in terms of an integral over x.

probability distribution: $|\psi_5(x)|^2 = \frac{2}{a}\sin^2\frac{5\pi x}{a}$

possible values: $0 < x < a$ (except not $\frac{a}{5}, \frac{2a}{5}, \frac{3a}{5}, \frac{4a}{5}$)

total probability $= 1$: $\int_{-\infty}^{\infty} dx\, |\psi_5(x)|^2 = \int_0^a \frac{2}{a}\sin^2\frac{5\pi x}{a} = 1$

(H) Suppose a large number N of particles are all in the $n = 5$ energy level and measurements of their positions give $x_1, x_2, \ldots x_N$. The average position is $\bar{x} = (x_1 + x_2 + \ldots + x_N)/N$. As N becomes larger and larger, what should you expect $\bar{x}$ to converge to? (Express it as an integral over x.)

as $N \to \infty$, $\bar{x} \longrightarrow \langle x\rangle = \int_{-\infty}^{\infty} dx\, x\,|\psi_5(x)|^2$

$= \int_0^a dx\, \frac{2}{a}\sin^2\frac{5\pi x}{a}$

Problem 3.

Consider the scattering of a particle in 1 dimension from a step potential that changes from 0 for $x < 0$ to V_0 for $x > 0$.

(A) Suppose a particle has energy E that is larger than V_0. What is its quantum wavelength in the region $x < 0$? What is its quantum wavelength in the region $x > 0$?

quantum wavelength: $\lambda = \frac{h}{p}$, where p is momentum

$X < 0$: $E = \frac{p^2}{2m} \Rightarrow p = \sqrt{2mE} \Rightarrow \lambda = \frac{h}{\sqrt{2mE}}$

$X > 0$: $E = \frac{p^2}{2m} + V_0 \Rightarrow p = \sqrt{2m(E-V_0)} \Rightarrow \lambda = \frac{h}{\sqrt{2m(E-V_0)}}$

The scattering probabilities for a particle of energy E can be determined from the solution of the time-independent Schroedinger equation for the wave function $\psi(x)$.

(B) Write down the most general solution to the Schroedinger equation for energy E in the region $x < 0$. (Define any symbols you introduce.)

$\psi(x) = Ae^{ikx} + Be^{-ikx}$

$-\frac{\hbar^2}{2m}(\pm ik)^2 = E \Rightarrow k = \frac{\sqrt{2mE}}{\hbar}$

(C) Write down the most general solution to the Schroedinger equation for energy E in the region $x > 0$.

$\psi(x) = Ce^{ik'x} + De^{-ik'x}$

$-\frac{\hbar^2}{2m}(\pm ik')^2 + V_0 = E \Rightarrow k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

Now consider the specific problem of a particle that comes from the left and scatters from the step potential.

(D) Identify the terms in the wavefunctions in parts B and C that correspond to incident, reflected and transmitted waves.

incident wave: Ae^{ikx}

reflected wave: Be^{-ikx}

transmitted wave: $Ce^{ik'x}$

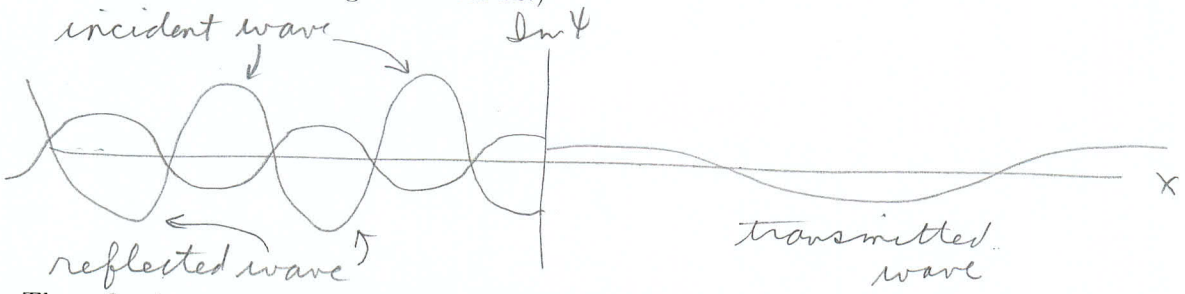
(E) Write down (but don't solve) the matching conditions for the wavefunction at $x = 0$. (They should be expressed as linear equations for coefficients in the general solutions to the Schroedinger equation.)

$\psi(0^-) = \psi(0^+)$: $A + B = C + D$

$\psi'(0^-) = \psi'(0^+)$: $ikA - ikB = ik'C - ik'D$

(but $D = 0$ if no particles coming from left)

(F) For the specific energy $\frac{4}{3}V_0$, the wavelength for $x > 0$ is twice the wavelength for $x < 0$. Sketch the real parts of the wavefunctions for the incident, reflected and transmitted waves as functions of x. (The amplitudes and phases of the waves do not need to be drawn accurately, but their relative wavelengths should be.)



The reflection and transmission probabilities for a particle of energy E are

$$R = \frac{(\sqrt{E} - \sqrt{E - V_0})^2}{(\sqrt{E} + \sqrt{E - V_0})^2}, \qquad T = \frac{4\sqrt{E}\sqrt{E - V_0}}{(\sqrt{E} + \sqrt{E - V_0})^2}.$$

They satisfy $R + T = 1$. (These equations for R and T are intended only to make the problem more concrete. You will not need to use them.)

(G) Express the probabilities R and T in terms of the amplitudes of the incident, reflected, and transmitted waves in parts B and C.

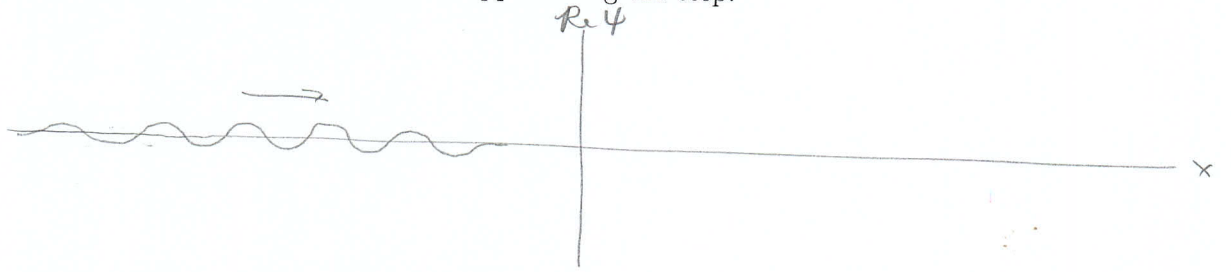
wave speeds $\quad x < 0: \; v = \sqrt{E/2m}$

$x > 0: \; v' = \sqrt{(E - V_0)/2m}$

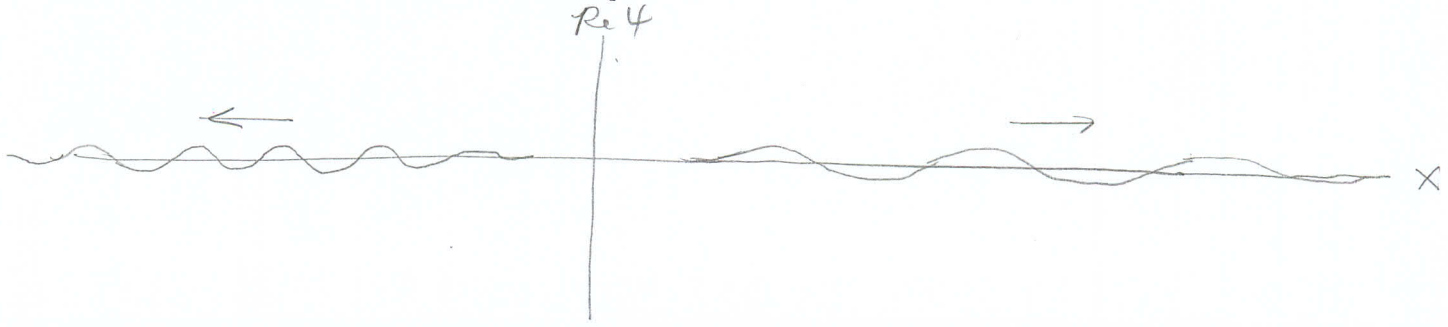
$$R = \frac{|B|^2 v}{|A|^2 v} = \frac{|B|^2}{|A|^2} \qquad T = \frac{|C|^2 v'}{|A|^2 v} = \frac{|C|^2}{|A|^2}\sqrt{\frac{E - V_0}{E}}$$

The scattering of a particle is actually described by a solution $\Psi(x, t)$ to the time-dependent Schroedinger equation. The wavefunction is a wavepacket consisting of waves with a narrow distribution of momenta that is sharply peaked near $\sqrt{2mE}$.

(H) For the specific energy $\frac{4}{3}V_0$ (see part F), sketch the real part of the wavefunction $\Psi(x, t)$ at a time t when the particle is still approaching the step.



(I) For the specific energy $\frac{4}{3}V_0$, sketch the real part of the wavefunction $\Psi(x, t)$ at a time t well after the particle has scattered from the step.



Suppose detectors are set up to observe the scattered particles: a LEFT detector far to the left to observe particles that are reflected and a RIGHT detector far to the right to observe particles that are transmitted.

(J) If a single particle with mass m and kinetic energy E is scattered from the step potential, which of the following statements describes what would be observed in the two detectors? Circle the numbers for each of the correct statements.

1. A particle will be observed either at the LEFT detector or at the RIGHT detector but not at both.
2. A particle will be observed at both the LEFT detector and the RIGHT detector.
3. A particle will be observed at the LEFT detector with probability R and at the RIGHT detector with probability T.
4. A particle with energy RE will be observed at the LEFT detector and a particle with energy TE will be observed at the RIGHT detector.
5. A particle with mass Rm will be observed in the LEFT detector and a particle with mass Tm will be observed in the RIGHT detector.

(K) Suppose a beam consisting of 1000 particles of mass m and kinetic energy E is scattered from the step potential. Approximately how many particles would be observed at the LEFT detector and how many at the RIGHT detector?

LEFT detector: $\approx 1000R$ particles

RIGHT detector: $\approx 1000T$ particles