Physics 5500 Final Exam 2012 NAME ______________________________

Problem 1.

If a spin-0 particle decays into two spin-$\frac{1}{2}$ particles A and B with no orbital angular momentum, their spins are entangled in a spin-singlet state:

$$\frac{1}{\sqrt{2}}|\uparrow\rangle_A|\downarrow\rangle_B - \frac{1}{\sqrt{2}}|\downarrow\rangle_A|\uparrow\rangle_B.$$

(A) Suppose the spin component S_{Az} is measured. List the possible values of S_{Az}, the probability for each value, and the spin state for particles A and B immediately after the measurement.

values of S_z	probability	spin state after
$+\frac{1}{2}\hbar$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$	$\lvert\uparrow\rangle_A \lvert\downarrow\rangle_B$
$-\frac{1}{2}\hbar$	$\left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$	$\lvert\downarrow\rangle_A \lvert\uparrow\rangle_B$

(B) Suppose the spin components S_{Az} and S_{Bz} are both measured. List the possible values of S_{Az} and S_{Bz} and the probability for each pair of values.

S_{Az}	S_{Bz}	probability
$+\frac{1}{2}\hbar$	$+\frac{1}{2}\hbar$	0
$+\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$
$-\frac{1}{2}\hbar$	$+\frac{1}{2}\hbar$	$\left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$
$-\frac{1}{2}\hbar$	$-\frac{1}{2}\hbar$	0

(C) What are the expectation values $\langle S_{Az}\rangle$ and $\langle S_{Az}S_{Bz}\rangle$.

$$\langle S_{Az}\rangle = \left(+\tfrac{1}{2}\hbar\right)\tfrac{1}{2} + \left(-\tfrac{1}{2}\hbar\right)\cdot\tfrac{1}{2} = 0$$

$$\langle S_{Az}S_{Bz}\rangle = \left(\tfrac{1}{2}\hbar\right)\left(\tfrac{1}{2}\hbar\right)\cdot 0 + \left(\tfrac{1}{2}\hbar\right)\left(-\tfrac{1}{2}\hbar\right)\cdot\tfrac{1}{2} + \left(-\tfrac{1}{2}\hbar\right)\left(\tfrac{1}{2}\hbar\right)\cdot\tfrac{1}{2} + \left(-\tfrac{1}{2}\hbar\right)\left(-\tfrac{1}{2}\hbar\right)\cdot 0$$
$$= -\tfrac{1}{4}\hbar^2$$

Bell's inequality states that if there are hidden variables that make the measurements deterministic, the expectation values of products of spin variables must satisfy

$$\left|\langle \hat{n}_1\cdot\vec{S}_A\,\hat{n}_2\cdot\vec{S}_B\rangle - \langle \hat{n}_1\cdot\vec{S}_A\,\hat{n}_3\cdot\vec{S}_B\rangle\right| \leq \frac{\hbar^2}{4} + \langle \hat{n}_2\cdot\vec{S}_A\,\hat{n}_3\cdot\vec{S}_B\rangle.$$

for any 3 unit vectors $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$.

(D) Show that for the spin-singlet state, quantum mechanics is compatible with Bell's inequality for the special case $\hat{n}_1 = \hat{n}_2 = \hat{n}_3 = \hat{z}$. (There are unit vectors for which they are not compatible.)

for $\hat{n}_1 = \hat{n}_2 = \hat{n}_3 = \hat{z}$, the inequality reduces to

$$\left|\langle S_{Az}S_{Bz}\rangle - \langle S_{Az}S_{Bz}\rangle\right| \leq \frac{\hbar^2}{4} + \langle S_{Az}S_{Bz}\rangle$$

$$\Longrightarrow \langle S_{Az}S_{Bz}\rangle \geq -\frac{\hbar^2}{4}$$

The only possible values of $S_{Az}S_{Bz}$ are $-\frac{1}{4}\hbar^2$ and $+\frac{1}{4}\hbar^2$. Thus its expectation value is greater than $-\frac{1}{4}\hbar^2$.

A particle could also decay into the two spin-$\frac{1}{2}$ particles A and B in a spin-triplet state, such as

$$\frac{1}{\sqrt{2}}|\uparrow\rangle_A|\downarrow\rangle_B + \frac{1}{\sqrt{2}}|\downarrow\rangle_A|\uparrow\rangle_B.$$

(E) List all the independent spin-triplet states and give the spin quantum numbers s and m_s for each of them.

state	s	m_s
$\lvert\uparrow\rangle_A \lvert\uparrow\rangle_B$	1	+1
$\frac{1}{\sqrt{2}}\left(\lvert\uparrow\rangle_A \lvert\uparrow\rangle_B + \lvert\downarrow\rangle_A \lvert\uparrow\rangle_B\right)$	1	0
$\lvert\downarrow\rangle_A \lvert\downarrow\rangle_B$	1	-1

The total spin for the two spin-$\frac{1}{2}$ particles is $\vec{S} = \vec{S}_A + \vec{S}_B$. If the particles have orbital angular momentum $\vec{L}$, the total angular momentum is $\vec{J} = \vec{L} + \vec{S}$. By conservation of angular momentum, $\vec{J}$ is also the angular momentum of the decaying particle.

(F) If the two particles are in a spin-triplet state and their orbital angular momentum quantum number is $\ell = 2$, what are the possible values of the angular momentum quantum number j for the decaying particle?

$l = 2, s = 1$

$j = |l-s|, |l-s|+1, \dots, l+s$

$= 1, 2, 3$

Supppose the particles A and B and a third spin-$\frac{1}{2}$ particle C have the spin state

$$\left(\frac{1}{\sqrt{2}}|\uparrow\rangle_A|\downarrow\rangle_B + \frac{1}{\sqrt{2}}|\downarrow\rangle_A|\uparrow\rangle_B\right)\left(\cos(\theta/2)e^{-i\phi/2}|\uparrow\rangle_C + \sin(\theta/2)e^{+i\phi/2}|\downarrow\rangle_C\right).$$

(Quantum teleportation would require further entanglement of the spins of particles B and C.)

(G) Suppose the spin components S_{Bz} and S_{Cz} are measured. What is the probability for obtaining $+\frac{1}{2}\hbar$ for S_{Bz} and $+\frac{1}{2}\hbar$ for S_{Cz}?

The component with $S_{Bz} = +\frac{1}{2}\hbar$ and $S_{Cz} = +\frac{1}{2}\hbar$ is

$\frac{1}{\sqrt{2}} |\downarrow\rangle_A |\uparrow\rangle_B \left(\cos\frac{\theta}{2} e^{-i\phi/2} |\uparrow\rangle_C\right) = \frac{1}{\sqrt{2}} \cos\frac{\theta}{2} e^{-i\phi/2} |\downarrow\rangle_A |\uparrow\rangle_B |\uparrow\rangle_C$

Its probability is $\left|\frac{1}{\sqrt{2}} \cos\frac{\theta}{2} e^{-i\phi/2}\right|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$

(H) Suppose measurements of the spin components S_{Bz} and S_{Cz} give $+\frac{1}{2}\hbar$ for S_{Bz} and $-\frac{1}{2}\hbar$ for S_{Cz}. What is the spin state of the three particles after the measurement?

The component with $S_{Bz} = +\frac{1}{2}\hbar$ and $S_{Cz} = -\frac{1}{2}\hbar$ is

$\frac{1}{\sqrt{2}} |\downarrow\rangle_A |\uparrow\rangle_B \left(\sin\frac{\theta}{2} e^{+i\phi/2} |\downarrow\rangle_C\right) = \frac{1}{\sqrt{2}} \sin\frac{\theta}{2} e^{-i\phi/2} |\downarrow\rangle_A |\uparrow\rangle_B |\downarrow\rangle_C$

The measurement collapses the spin state to

$|\downarrow\rangle_A |\uparrow\rangle_B |\downarrow\rangle_C$

Problem 2.

A particle of mass m is confined in a cubic box of volume L^3 by a potential that is 0 inside the box and $+\infty$ outside. The energy eigenvalues and the corresponding normalized wavefunctions can be labelled by quantum numbers n_x, n_y, and n_z:

$$\begin{aligned} E_{n_x,n_y,n_z} &= (n_x^2+n_y^2+n_z^2)h^2/(8mL^2), \\ \psi_{n_x,n_y,n_z}(x,y,z) &= (8/L^3)^{1/2}\sin(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L). \end{aligned}$$

(A) These energies and wavefunctions are the solutions to an eigenvalue problem. Express this eigenvalue problem as a partial differential equation with boundary conditions.

$$-\frac{\hbar^2}{2m}\left[\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2+\left(\frac{\partial}{\partial z}\right)^2\right]\psi(x,y,z) = E\,\psi(x,y,z)$$

$$\psi(x=0,y,z)=0 \qquad \psi(x,y=0,z)=0 \qquad \psi(x,y,z=0)=0$$

$$\psi(x=L,y,z)=0 \qquad \psi(x,y=L,z)=0 \qquad \psi(x,y,z=L)=0$$

(B) Suppose the single particle inside the cube is in the ground state. What is its energy? If the position of the particle is measured, what is the probability density for the coordinates x, y, and z?

$$E_{1,1,1} = 3\frac{h^2}{8mL^2}$$

$$|\psi_{111}(x,y,z)|^2 = \frac{8}{L^3}\sin^2\frac{\pi x}{L}\sin^2\frac{\pi y}{L}\sin^2\frac{\pi z}{L}$$

(C) Suppose two identical spin-0 bosons are confined inside the cube. If the 2-particle system is in the ground state, what are the possible quantum numbers for the orbitals of the two bosons? What is the total energy?

orbitals: $(n_x,n_y,n_z) = (1,1,1)$ and $(1,1,1)$

energy: $2E_{111} = \frac{3h^2}{4mL^2}$

(D) Suppose two identical fermions in the same spin state are confined inside the cube. If the 2-particle system is in the ground state, what are the possible quantum numbers for the orbitals of the two fermions? What is the total energy?

orbitals: $(n_x,n_y,n_z) = (1,1,1)$ and $(2,1,1)$
$(1,1,1)$ and $(1,2,1)$
$(1,1,1)$ and $(1,1,2)$

energy: $E_{111}+E_{211} = \frac{9h^2}{8mL^2}$

(E) Suppose a large number N of identical spin-0 bosons are confined inside the cube. If the N-particle system is in the ground state, what are the quantum numbers for the orbitals of the N bosons? What is the total energy of the bosons?

The N orbitals are all $(1,1,1)$

Total energy: $NE_{111} = N\frac{3h^2}{8mL^2}$

Suppose a large number N of identical fermions all in the same spin state are confined inside the cube. Since most of the fermions have large quantum numbers, their orbitals can be labelled by the position (x, y, z) and the momentum (p_x, p_y, p_z). The orbitals can be counted by integrating over these 6 variables using the measure $\int dxdydz \int dp_xdp_ydp_z/h^3$.

(F) What is the energy ε of an orbital labelled by the 6 variables (x, y, z, p_x, p_y, p_z)? What is the length p of the momentum vector for the orbital?

$$\epsilon = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

$$p = \sqrt{p_x^2 + p_y^2 + p_z^2}$$

(G) In the ground state, the fermions occupy all orbitals with energy less than or equal to the Fermi energy ε_F. What is the position integral $\int dxdydz$ for the occupied orbitals? What is the momentum integral $\int dp_xdp_ydp_z$ for the occupied orbitals?

The cube has volume $\int dx\,dy\,dz = L^3$

The occupied region of momentum space is a sphere of radius p_F, where $\epsilon_F = \frac{1}{2m}p_F^2$:

$$\int dp_x\,dp_y\,dp_z = \frac{4}{3}\pi p_F^3 = \frac{4}{3}\pi(2m\epsilon_F)^{3/2}$$

(H) Use the condition that the number N of fermions is equal to the number of occupied orbitals to deduce the Fermi energy ε_F as a function of N and L.

$$N = \text{number of occupied orbitals}$$

$$= \frac{1}{h^3}\int dx\,dy\,dz \int dp_x\,dp_y\,dp_z$$

$$= \frac{1}{h^3} \cdot L^3 \cdot \frac{4}{3}\pi(2m\epsilon_F)^{3/2}$$

Solve for ϵ_F: $(2m\epsilon_F)^{3/2} = \frac{3}{4\pi}h^3\frac{N}{L^3}$

$$\epsilon_F = \frac{1}{2m}h^2\left(\frac{3}{4\pi}\frac{N}{L^3}\right)^{2/3}$$

Suppose the potential energy outside the cube is $+V_0$ instead of $+\infty$, with $V_0 > \varepsilon_F$. The N-fermion system is in the ground state, with all orbitals with energy less than ε_F occupied. Light of frequency ν is shining into the cube.

(I) What is the range of energies for a fermion in the box that has absorbed a single photon?

range of energies before absorbing the photon: $0 < E < \epsilon_F$

after " " : $hf < E < \epsilon_F + hf$

(J) What is the minimum frequency ν for which a fermion can escape from the cube by absorbing a single photon?

maximum energy after absorbing the photon must be greater than the potential energy outside the box

$$\epsilon_F + hf > V_0$$

minimum frequency: $\epsilon_F + hf = V_0$ $\quad f = \dfrac{V_0 - \epsilon_F}{h}$

Problem 3.

The Hamiltonian for a particle in a one-dimensional potential is $H = H_0 + H_1$, where

$$H_0 = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2,$$
$$H_1 = KX^4.$$

The eigenstates $|n\rangle$ of H_0 can be labelled by a quantum number $n = 0, 1, 2, \ldots$. The eigenvalues of H_0 and the normalized eigenfunctions are

$$E_n = (n + \tfrac{1}{2})\hbar\omega,$$
$$\psi_n(x) = H_n(\beta x)\exp(-\beta^2 x^2/2),$$

where $\beta = \sqrt{m\omega/\hbar}$ and $H_n(z)$ is a polynomial of degree n.

Suppose the coefficient K is small (compared to $m\omega^2\beta^2$), so that H_1 can be treated as a perturbation. If the eigenvalues and eigenfunctions of H are expanded in powers of K, the eigenvalue equation is

$$(H_0 + H_1)\left(|n\rangle + |\psi_n^{(1)}\rangle + |\psi_n^{(2)}\rangle + \ldots\right)$$
$$= \left(E_n + E_n^{(1)} + E_n^{(2)} + \ldots\right)\left(|n\rangle + |\psi_n^{(1)}\rangle + |\psi_n^{(2)}\rangle + \ldots\right).$$

(A) The eigenvalue equation can be expanded in powers of K. Write down the equations that are obtained at 0^{th} order in K and at 1^{st} order in K.

0^{th} order: $H_0|n\rangle = E_n|n\rangle$

1^{st} order: $H_1|n\rangle + H_1|\psi_n^{(1)}\rangle = E_n^{(1)}|n\rangle + E_n|\psi_n^{(1)}\rangle$

(B) Express the first-order correction $E_n^{(1)}$ to the n$^{\text{th}}$ energy eigenvalue as an integral over x.

$$E_n^{(1)} = \langle H_1\rangle_n = \frac{\int dx\, \psi_n^*(x)\, Kx^4 \psi_n(x)}{\int dx\, \psi_n^*(x)\, \psi_n(x)}$$

$$= K\int_{-\infty}^{\infty} dx\, x^4\, H_n(\beta x)^2 e^{-\beta^2 x^2}$$

The second-order correction $E_n^{(2)}$ to the n$^{\text{th}}$ energy eigenvalue can be expressed as

$$E_n^{(2)} = \sum_m \frac{\langle n|H_1|m\rangle\langle m|H_1|n\rangle}{E_n - E_m}.$$

(C) What are the values of m that must be summed over? Express the energy denominator $E_m - E_n$ in as simple a form as possible. Express the matrix element $\langle m|H_1|n\rangle$ as an integral over x.

$m = 0, 1, 2, \ldots$ but excluding n

$$E_m - E_n = (m + \tfrac{1}{2})\hbar\omega - (n + \tfrac{1}{2})\hbar\omega = (m-n)\hbar\omega$$

$$\langle m|H_1|n\rangle = \int_{-\infty}^{\infty} dx\, \psi_m^*(x)\, Kx^4\, \psi_n(x)$$

$$= K\int_{-\infty}^{\infty} dx\, x^4\, H_m(\beta x) H_n(\beta x)\, e^{-\beta^2 x^2}$$

Suppose K is large (compared to $m\omega^2\beta^2$). A simple variational wavefunction for the ground state is $\psi(x) = \exp(-\alpha x^2/2)$, where α is an adjustable parameter.

(D) Express the norm of the wavefunction as an integral over x.

$$\|\psi\|^2 = \int dx\, |\psi(x)|^2 = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}$$

(E) Express the variational estimate $E_0(\alpha)$ for the ground state energy in terms of integrals over x.

$$E_0(\alpha) = \langle H \rangle_\psi$$

$$= \frac{\int dx\, \psi^*(x)\left[-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 + \frac{1}{2}m\omega^2 x^2 + Kx^4\right]\psi(x)}{\int dx\, \psi^*(x)\psi(x)}$$

$$= \frac{\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2/2}\left[-\frac{\hbar^2}{2m}\left(\frac{d}{dx}\right)^2 + \frac{1}{2}m\omega^2 x^2 + Kx^4\right]e^{-\alpha x^2/2}}{\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}}$$

If the integrals over x are evaluated, the variational approximation to the ground state energy reduces to

$$E_0(\alpha) = \frac{\hbar^2\alpha}{2m} + \frac{m\omega^2}{4\alpha} + \frac{3K}{4\alpha^2}.$$

(F) There is one value of α that gives the best possible estimate for the ground state energy. What polynomial equation does this optimal value of α satisfy?

$E_0(\alpha)$ is a minimum at the value of α that satisfies

$$\frac{d}{d\alpha}E_0(\alpha) = 0$$

$$\frac{\hbar^2}{2m} + \frac{m\omega^2}{4}\left(-\frac{1}{\alpha^2}\right) + \frac{3K}{4}\left(-\frac{2}{\alpha^3}\right) = 0$$

multiply by α^3:

$$\frac{\hbar^2}{2m}\alpha^3 - \frac{m\omega^2}{4}\alpha - \frac{3K}{2} = 0$$

Suppose the Hamiltonian $H_1(t)$ is time dependent, with a coefficient K that is 0 for $t < 0$ and a constant K_0 for $t > 0$. The normalized wavefunction for $t > 0$ can be expanded in eigenstates of H_0 as follows:

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t) \exp(i(E_n - E_0)t/\hbar)\, |n\rangle.$$

(G) Suppose that at the positive time T, the perturbation H_1 is turned off (so the Hamiltonian is just H_0) and the energy is measured. What is the probability that the measured value is E_n? What is the quantum state $|\psi(T^+)\rangle$ immediately after a measurement that gives the value E_n?

probability : $\left| C_n(t)\, e^{i(E_n - E_0)t/\hbar} \right|^2 = |C_n(t)|^2$

$|\psi(T^+)\rangle = |n\rangle$

(H) If the particle is in the ground state for $t < 0$, the solution for the coefficient $c_n(t)$ for $n > 0$ to first order in perturbation theory is

$$c_n(t) = \frac{1}{i\hbar} \int_0^t dt' \exp(i(E_n - E_0)t'/\hbar) \langle n|H_1(t')|0\rangle.$$

Express this amplitude as the product of an integral over x and an integral over t'.

$$C_n(t) = \frac{1}{i\hbar} \int_0^t dt'\, e^{i(E_n - E_0)t'/\hbar} \int dx\, \psi_n^*(x)\, K_0\, \psi_0(x)$$

$$= \frac{K_0}{i\hbar} \int_0^t dt'\, e^{i(E_n - E_0)t'/\hbar} \int_{-\infty}^{\infty} dx\, H_n(\beta x) H_0(\beta x) e^{-\beta^2 x^2}$$