Ultracold Trapped Atoms

N identical atoms
all in the same hyperfine spin state
trapped in a 3-D harmonic oscillator potential

$$V(r) = \tfrac{1}{2} m \omega^2 r^2$$

$$= \tfrac{1}{2} m \omega^2 (x^2 + y^2 + z^2)$$

Hamiltonian

$$H = \sum_{i=1}^{N} \left(\frac{1}{2m} \vec{P}_i^{\,2} + V(r_i) \right)$$

single particle quantum states (orbitals)

labelled by 3 quantum numbers: $|n_x, n_y, n_z\rangle$

$n_x, n_y, n_z = 0, 1, 2, \ldots$

orbital energies: $\epsilon_{n_x, n_y, n_z} = (n_x + n_y + n_z + \tfrac{3}{2}) \hbar \omega$

N-particle quantum states: specified by N orbitals

$$|n_{1x}, n_{1y}, n_{1z} ; n_{2x}, n_{2y}, n_{2z} ; \ldots ; n_{Nx}, n_{Ny}, n_{Nz}\rangle$$

$\begin{pmatrix} \text{symmetrize} \\ \text{anti-symmetrize} \end{pmatrix}$ if atom is a $\begin{pmatrix} \text{boson} \\ \text{fermion} \end{pmatrix}$

^7Li atoms (bosons)

ground state: all atoms in lowest energy orbital

lowest energy orbital: $|0,0,0\rangle$

energy: $\epsilon_{000} = \frac{3}{2}\hbar\omega$

wavefunction: $\psi_{000}(x,y,z) = \sqrt{\frac{\beta}{\sqrt{\pi}}}\, e^{-\beta^2 x^2/2} \cdot \sqrt{\frac{\beta}{\sqrt{\pi}}}\, e^{-\beta^2 y^2/2} \cdot \sqrt{\frac{\beta}{\sqrt{\pi}}}\, e^{-\beta^2 z^2/2}$

$$= \left(\frac{\beta}{\sqrt{\pi}}\right)^{3/2} e^{-\beta^2 r^2/2}$$

$$\beta = \sqrt{\frac{m\omega}{\hbar}}$$

all N atoms have same wavefunction with same phase
"Bose-Einstein condensate"

number density of atom

$$n(r) = N|\psi_{000}(r)|^2$$

$$= N\left(\frac{\beta}{\sqrt{\pi}}\right)^3 e^{-\beta^2 r^2}$$ Gaussian with width $\frac{1}{\beta}$

$$\int d^3r\, n(r) = N\left(\frac{\beta}{\sqrt{\pi}}\right)^3 4\pi \int_0^\infty r^2 dr\, e^{-\beta^2 r^2}$$

$$= N\left(\frac{\beta}{\sqrt{\pi}}\right)^3 4\pi \frac{\sqrt{\pi}}{4\beta^3} = N$$

^{6}Li atoms (fermions)

ground state: one atom in each of
N lowest-energy orbitals

most atoms have large quantum numbers

$\Longrightarrow$ orbitals can be labelled by
classical phase space variables
position: $\vec{r} = (x, y, z)$
momentum $\vec{p} = (p_x, p_y, p_z)$

orbital energy: $\epsilon(r, p) = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 r^2$

counting of orbitals: $\int \frac{d^3r\, d^3p}{(2\pi\hbar)^3}$

ground state: all orbitals with $\epsilon(r,p) < \epsilon_F$ occupied

Fermi energy ϵ_F to be determined

$\frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 r^2 < \epsilon_F$

$\Longrightarrow p < p_F(r) = \sqrt{2m(\epsilon_F - \frac{1}{2} m \omega^2 r^2)}$ Fermi momentum at radius r

$\Longrightarrow r < R_F = \sqrt{\frac{2\epsilon_F}{m\omega^2}}$ Fermi radius number of atoms = number of occupied orbital

$$N = \int_{\epsilon(r,p)<\epsilon_F} \frac{d^3r\, d^3p}{(2\pi\hbar)^3}\, 1$$

$$= \frac{1}{(2\pi\hbar)^3} \int_{r<R_F} d^3r \int_{p<p_F(r)} d^3p\, 1$$

$$= \frac{1}{(2\pi\hbar)^3}\, 4\pi \int_0^{R_F} r^2 dr\, 4\pi \underbrace{\int_0^{p_F(r)} p^2 dp}_{\frac{1}{3}p_F(r)^3}$$

$$\epsilon_F = \frac{1}{2} m\omega^2 R_F^2 \implies p_F = \sqrt{2m(\epsilon_F - \frac{1}{2}m\omega^2 r^2)}$$
$$= m\omega\sqrt{R_F^2 - r^2}$$

$$N = \frac{(4\pi)^2}{(2\pi\hbar)^3} \int_0^{R_F} r^2 dr\, \frac{1}{3}\left(m\omega\sqrt{R_F^2-r^2}\right)^3$$

$$= \frac{2}{3\pi}\left(\frac{m\omega}{\hbar}\right)^3 \underbrace{\int_0^{R_F} dr\, r^2 (R_F^2 - r^2)^{3/2}}_{\frac{\pi}{32}R_F^6}$$

$$= \frac{1}{48}\beta^6 R_F^6$$

Fermi radius : $R_F = (48N)^{1/6}\frac{1}{\beta}$ $\quad \beta = \sqrt{\frac{m\omega}{\hbar}}$

Fermi energy : $\epsilon_F = \frac{1}{2}m\omega^2 R_F^2$

$$= (48N)^{1/3}\, \frac{1}{2}\hbar\omega$$

number density of atoms: $n(r)$

$$N = \frac{1}{6\pi^2}\beta^6 \cdot 4\pi \int_0^{R_F} dr\, r^2 \left(R_F^2 - r^2\right)^{3/2}$$

$$= 4\pi \int_0^\infty r^2 dr\, n(r)$$

$$n(r) = \frac{1}{6\pi^2}\beta^6 \left(R_F^2 - r^2\right)^{3/2} \qquad r < R_F$$

$$= 0 \qquad r > R_F$$

$\Rightarrow$ sharp edge at $r = R_F = (48N)^{1/6}\frac{1}{\beta}$