Expectation Values for Spin $\frac{1}{2}$

spin operators for spin-$\frac{1}{2}$

$$S_x = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \tfrac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

eigenstates of S_z

$S_z = +\frac{1}{2}\hbar$: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ spin up (along z direction)

$S_z = -\frac{1}{2}\hbar$: $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ spin down (along z direction)

expectation values of spin operators

$$\langle\uparrow|S_z|\uparrow\rangle = \langle\uparrow|\left(+\tfrac{1}{2}\hbar|\uparrow\rangle\right) = \tfrac{1}{2}\hbar\langle\uparrow|\uparrow\rangle = \tfrac{1}{2}\hbar$$

$$\langle\uparrow|S_x|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger \tfrac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\langle\uparrow|S_y|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger \tfrac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ i \end{pmatrix} = 0$$

$$\langle\uparrow|\overline{S}|\uparrow\rangle = \tfrac{1}{2}\hbar\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \tfrac{1}{2}\hbar\,\hat{z}$$

most general normalized spin state

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\uparrow\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where α, β are complex constants
satisfying $|\alpha|^2 + |\beta|^2 = 1$

$$|\psi\rangle = e^{i\chi/2}\left(\cos\tfrac{\theta}{2}\, e^{-i\phi/2}|\uparrow\rangle + \sin\tfrac{\theta}{2}\, e^{+i\phi/2}|\downarrow\rangle\right)$$

where χ, θ, ϕ are real constants

expectation values of spin operators

$$\langle\psi|S_z|\psi\rangle = \begin{pmatrix} e^{i\chi/2}\cos\frac{\theta}{2}\, e^{-i\phi/2} \\ e^{i\chi/2}\sin\frac{\theta}{2}\, e^{+i\phi/2} \end{pmatrix}^{\dagger} \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{i\chi/2}\cos\frac{\theta}{2}\, e^{-i\phi/2} \\ e^{i\chi/2}\sin\frac{\theta}{2}\, e^{+i\phi/2} \end{pmatrix}$$

$$= \frac{1}{2}\hbar\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = \frac{1}{2}\hbar\cos\theta$$

$$\langle\psi|S_x|\psi\rangle = \frac{1}{2}\hbar\sin\theta\cos\phi$$

$$\langle\psi|S_y|\psi\rangle = \frac{1}{2}\hbar\sin\theta\sin\phi$$

$$\langle\psi|\vec{S}|\psi\rangle = \frac{1}{2}\hbar\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

$|\psi\rangle$ is spin up in the direction of angles θ, ϕ !

constant magnetic field along z-axis

$$\vec{B} = B_0 \hat{z}$$

magnetic moment of electron
(in state with 0 orbital angular momentum)

$$\vec{\mu} = -\frac{ge}{2m_e}\vec{S}$$

Hamiltonian for spin state of electron

$$H = -\vec{\mu}\cdot\vec{B}$$

$$= \frac{ge}{2m_e}\vec{S}\cdot\vec{B}$$

$$= \frac{ge}{2m_e}B_0 S_z$$

$$= \omega S_z \quad , \quad \text{where } \omega = \frac{geB_0}{2m_e}$$

Schroedinger equation for spin state

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle = \omega S_z|\psi\rangle$$

hermitian conjugate:

$$-i\hbar\frac{d}{dt}\langle\psi| = \langle\psi|H^\dagger = \langle\psi|H$$

expectation value of operator $\mathcal{O}$ in state $|\psi\rangle$

$$\langle \mathcal{O} \rangle_\psi = \langle \psi | \mathcal{O} | \psi \rangle$$

time evolution:

$$i\hbar \frac{d}{dt} \langle \mathcal{O} \rangle_\psi = \left(i\hbar \frac{d}{dt} \langle \psi | \right) \mathcal{O} | \psi \rangle + \langle \psi | \mathcal{O} \left(i\hbar \frac{d}{dt} | \psi \rangle \right)$$

$$= \left(- \langle \psi | H \right) \mathcal{O} | \psi \rangle + \langle \psi | \mathcal{O} \left(H | \psi \rangle \right)$$

$$= \langle \psi | \left(\mathcal{O} H - H \mathcal{O} \right) | \psi \rangle$$

$$= \langle [\mathcal{O}, H] \rangle_\psi$$

$$i\hbar \frac{d}{dt} \langle S_z \rangle_\psi = \omega \langle [S_z, S_z] \rangle_\psi$$

$$= 0$$

$$i\hbar \frac{d}{dt} \langle S_x \rangle_\psi = \omega \langle [S_x, S_z] \rangle_\psi$$

$$= \omega \langle -i\hbar S_y \rangle_\psi = -i\hbar\omega \langle S_y \rangle_\omega$$

$$i\hbar \frac{d}{dt} \langle S_y \rangle_\psi = \omega \langle [S_y, S_z] \rangle_\psi$$

$$= \omega \langle i\hbar S_x \rangle_\psi = i\hbar\omega \langle S_x \rangle_\psi$$

$$\frac{d}{dt}\langle S_x\rangle_4 = -\omega\langle S_y\rangle_4$$

$$\frac{d}{dt}\langle S_y\rangle_4 = +\omega\langle S_x\rangle_4$$

$$\frac{d}{dt}\langle S_z\rangle = 0$$

general solution

$$\langle S_x\rangle_4 = A\cos(\omega t+\phi)$$

$$\langle S_y\rangle_4 = A\sin(\omega t+\phi)$$

$$\langle S_z\rangle_4 = B$$

where A, B, ϕ are constants

$\langle\vec{S}\rangle_4$ precesses around z-axis
with angular frequency $\omega = \frac{geB_0}{2m_e}$

for general constant magnetic field $\vec{B}$,

$\langle\vec{S}\rangle_4$ precesses around $\vec{B}$
with angular frequency $\frac{ge}{2m_e}|\vec{B}|$