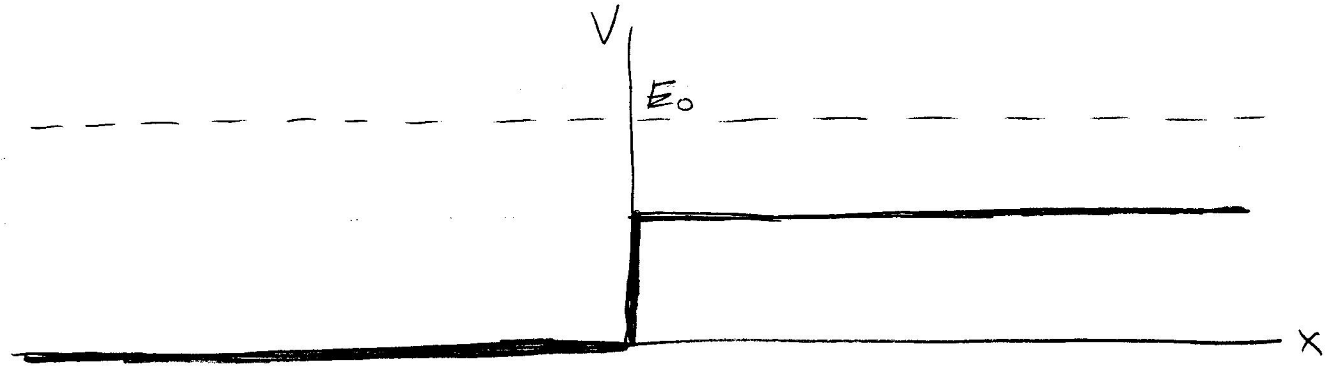
Scattering: Time-Independent Schroedinger Equation

step potential: $V(x) = 0 \quad x<0$
$\quad\quad\quad\quad\quad\quad\quad = V_0 \quad x>0$

particle with energy $E_0 > V_0$ approaching from left



time-independent Schroedinger equation

wavefunction: $\Psi(x,t) = \psi(x)\, e^{-iE_0 t/\hbar}$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 \psi + V(x)\,\psi = E_0 \psi$$

$$\begin{cases} -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 \psi + 0\cdot\psi = E_0 \psi & x<0 \\ -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 \psi + V_0 \psi = E_0 \psi & x>0 \end{cases}$$

simple solutions for $x < 0$

$$\psi(x) = e^{ikx} \text{ and } e^{-ikx}$$

$$-\frac{\hbar^2}{2m}(\pm ik)^2 \psi + 0\,\psi = E_0 \psi$$

$$\Rightarrow \frac{\hbar^2 k^2}{2m} = E_0 \quad \Rightarrow \quad k = \frac{\sqrt{2mE_0}}{\hbar}$$

simple solutions for $x > 0$

$$\psi(x) = e^{ik'x} \text{ and } e^{-ik'x}$$

$$-\frac{\hbar^2}{2m}(\pm ik')^2 \psi + V_0 \psi = E_0 \psi$$

$$\Rightarrow \frac{\hbar^2 k'^2}{2m} = E_0 - V_0 \quad \Rightarrow \quad k' = \frac{\sqrt{2m(E_0 - V_0)}}{\hbar}$$

solution for scattering problem

$$\psi(x) = A e^{ikx} + B e^{-ikx} \qquad x < 0$$

$$= C e^{ik'x} + 0 \cdot e^{-ik'x}$$

no incoming wave from left

boundary conditions at $x=0$

$$\psi(0^+) = \psi(0^-): \quad A + B = C$$

$$\psi'(0^+) = \psi'(0^-): \quad A(ik) + B(-ik) = C(ik')$$

2 linear equations in 3 unknowns (A, B, C)

$\Longrightarrow$ can solve for B, C in terms of A

$$B = \frac{k-k'}{k+k'} A$$

$$C = \frac{2k}{k+k'} A$$

solution for $x<0$:

$$\psi(x) = A e^{ikx}$$
probability per length $\propto |A|^2$ flowing to right with speed $\hbar k/m$

$$+ B e^{-ikx}$$
probability per length $\propto |B|^2$ flowing to left with speed $\hbar k/m$

solution for $x>0$:

$$\psi(x) = C e^{ik'x}$$
probability per length $\propto |C|^2$ flowing to right with speed $\hbar k'/m$

conservation of probability at $x=0$:

$$|A|^2 \left(\frac{\hbar k}{m}\right) + |B|^2 \left(-\frac{\hbar k}{m}\right) = |C|^2 \frac{\hbar k'}{m}$$

reflection probability:

R = ratio of probability flows in reflected wave and incoming wave

$$= \frac{|B|^2(\hbar k/m)}{|A|^2(\hbar k/m)} = \frac{|B|^2}{|A|^2}$$

$$= \frac{(k-k')^2}{(k+k')^2} = \frac{\left(1-\sqrt{(E-V_0)/E_0}\right)^2}{\left(1+\sqrt{(E-V_0)/E_0}\right)^2}$$

transmission probability

T = ratio of probability flows in transmitted wave and incoming wave

$$= \frac{|C|^2(\hbar k'/m)}{|A|^2(\hbar k/m)} = \frac{|C|^2}{|A|^2}\frac{k'}{k}$$

$$= \frac{2kk'}{(k+k')^2} = \frac{2\sqrt{(E-V_0)/E_0}}{\left(1+\sqrt{(E-V_0)/E_0}\right)^2}$$

conservation of probability

$$R+T=1$$

Note: On page 72, Serway's equation for T implies $T = |C|^2/|A|^2$.
This equation is wrong.