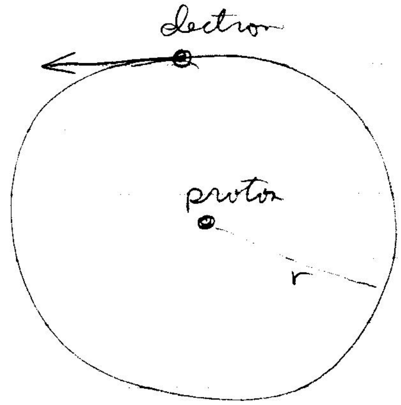
Hydrogen Atom

classical model: electron orbits proton
like planet around sun



electron: electric charge $-e$
mass m_e

proton: electric charge $+e$
mass $\gg m_e$

simple circular orbit: radius r
velocity v
acceleration $a = \frac{v^2}{r}$

Newton's equation: $\vec{F} = m_e \vec{a}$

$$-\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r^2} = m_e\left(-\frac{v^2}{r}\right)$$

$$\Longrightarrow \quad v^2 = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{mr}, \quad a = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{m_e r^2}$$

Energy: $E = \frac{1}{2}m_e v^2 - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$

$$= -\frac{1}{2}m_e v^2$$

$$= -\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r}$$

Maxwell's equations for electromagnetism

accelerating charged particle
radiates electromagnetic waves

electric charge q

acceleration a

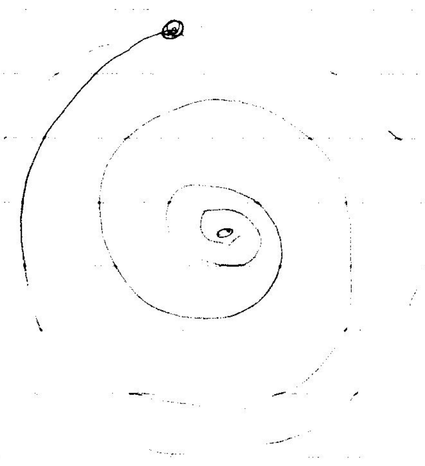
power radiated $P = \frac{2}{3}\left(\frac{q^2}{4\pi\epsilon_0}\right)\frac{a^2}{c^3}$

electron in circular orbit
radiates electromagnetic waves
whose frequency ν
is the same as orbital frequency $\frac{v}{2\pi r}$

acceleration: $a = \frac{v^2}{r} = \frac{1}{m_e}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r^2}$

power radiated $P = -\frac{dE}{dt}$

$\Longrightarrow$ energy E decreases
radius r decreases



electron orbit is a spiral
with decreasing radius!

hydrogen atom unstable!

Express E and $P = -\frac{dE}{dt}$ as functions of r only using $v^2 = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{m_e r}$

$$a = \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{m_e r^2}$$

$$E = -\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r}$$

$$P = \frac{2}{3}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{c^3}\left[\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{m_e r^2}\right]^2$$

$$= \frac{2}{3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^3\frac{1}{m_e^2 c^3 r^4}$$

differentiate E: $\frac{dE}{dt} = +\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r^2}\frac{dr}{dt}$

set $P = -\frac{dE}{dt}$: $\frac{2}{3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^3\frac{1}{m_e^2 c^3 r^4} = -\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r^2}\frac{dr}{dt}$

$$\underbrace{r^2\frac{dr}{dt}}_{\frac{d}{dt}\left(\frac{1}{3}r^3\right)} = -\frac{4}{3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{m_e^2 c^3}$$

solution: $r^3(t) = r^3(0) - 4\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{m_e^2 c^3}t$

electron crashes into proton at time T: $r(T) = 0$

$$T = \frac{m_e^2 c^3 r^3(0)}{4\left(\frac{e^2}{4\pi\epsilon_0}\right)^2}$$

Plug in size of hydrogen atom: $r(0) \sim 10^{-10}\,m$

$\Longrightarrow$ lifetime of hydrogen atom: $T \sim 10^{-7}\,s$

Hot hydrogen gas emits light at discrete wavelengths

Balmer (1885): fit to visible spectrum

$$\frac{1}{\lambda_n} = \frac{1}{\lambda_B}\left(\frac{1}{4} - \frac{1}{n^2}\right), \quad n = 3, 4, 5, \ldots$$

Balmer wavelength: $\lambda_B \approx 10^{-7} m$

Rydberg (1890): predict complete spectrum

$$\frac{1}{\lambda_{n,m}} = \frac{1}{\lambda_B}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \quad \begin{array}{l} m = 1, 2, 3, \ldots \\ n = m+1, m+2, \ldots \end{array}$$

$m=1$: ultraviolet
discovered by Lyman (1906)

$m=3$: infrared
discovered by Paschen (1908)

Discovery of electron : Thomson (1898)

" atomic nucleus : Rutherford (1909)

" proton : Rutherford (1917)

Rydberg spectrum for hydrogen atom

$$\frac{1}{\lambda_{nm}} = \frac{1}{\lambda_B}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

multiply by hc
frequency: $\nu = \frac{c}{\lambda}$

$$h\nu_{nm} = \frac{hc}{\lambda_B}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

$$= Ry\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

Rydberg energy: $Ry = 13.6\ \mathrm{eV}$

<u>Bohr model</u> of hydrogen atom (1913)

- electron in hydrogen atom
  has discrete energy levels

$$E_n = -\frac{Ry}{n^2}, \quad n = 1, 2, 3, \ldots$$

- transition from energy level n
  to " m
  gives packet of electromagnetic radiation
  with wavelength $\lambda_{n,m}$
  total energy $h\nu_{n,m}$

conservation of energy: $E_n = E_m + h\nu_{n,m}$

- electron orbits are circles

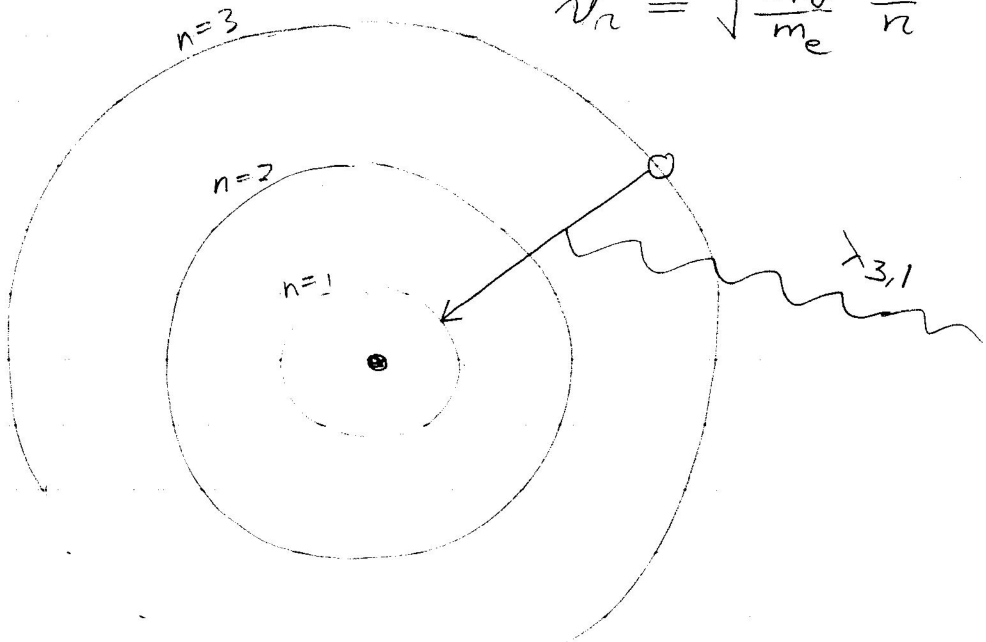
discrete energies $E_n = -\frac{Ry}{n^2}$

$\Longrightarrow$ discrete radii: $-\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{r_n} = -\frac{Ry}{n^2}$

$$r_n = \frac{1}{2Ry}\left(\frac{e^2}{4\pi\epsilon_0}\right)n^2$$

$\Longrightarrow$ discrete velocities: $-\frac{1}{2}m_e v_n^2 = -\frac{Ry}{n^2}$

$$v_n = \sqrt{\frac{2 Ry}{m_e}}\ \frac{1}{n}$$



combination of r_n and v_n

$$\frac{v_n}{r_n} = \frac{(2Ry)^{3/2}}{\sqrt{m_e}\,(e^2/4\pi\epsilon_0)}\frac{1}{n^3}$$

$$v_n r_n = \frac{1}{\sqrt{m_e \cdot 2Ry}}\left(\frac{e^2}{4\pi\epsilon_0}\right)n$$

- large quantum numbers n, m correspond to classical physics

large $n \Longrightarrow$ large r_n
small v_n

classical physics: electromagnetic radiation has frequency of electron orbit
$$\nu = \frac{v}{2\pi r}$$

$$h\nu_{m+1,m} = -R_y\left(\frac{1}{m^2} - \frac{1}{(m+1)^2}\right)$$

$$\longrightarrow h\frac{v_m}{2\pi r_m} \quad \text{as} \quad m \longrightarrow \infty$$

binomial expansion: $\frac{1}{(m+1)^2} = \frac{1}{\left[m\left(1+\frac{1}{m}\right)\right]^2}$

$$= \frac{1}{m^2}\left(1+\frac{1}{m}\right)^{-2}$$

$$\approx \frac{1}{m^2}\left(1-\frac{2}{m}\right)$$

$$= \frac{1}{m^2} - \frac{2}{m^3}$$

$$-R_y\left[\cancel{\frac{1}{m^2}} - \left(\cancel{\frac{1}{m^2}} - \frac{2}{m^3}\right)\right] \longrightarrow h\frac{1}{2\pi}\frac{(2R_y)^{3/2}}{\sqrt{m_e}\left(\frac{e^2}{4\pi\epsilon_0}\right)}\frac{1}{m^3}$$

dependence on m agrees
determines R_y in terms of fundamental constant

$$R_y = \frac{2\pi^2 m_e}{h^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2$$

angular momentum of electron
in circular orbit

$$L = m_e v r$$

discrete energy levels

$\Longrightarrow$ discrete angular momentum

$$L_n = m_e v_n r_n$$

$$= \sqrt{\frac{m_e}{2R_y}}\left(\frac{e^2}{4\pi\epsilon_0}\right) n$$

$$= \frac{1}{2\pi} h n$$

$$= n \hbar \qquad \hbar \equiv \frac{h}{2\pi}$$

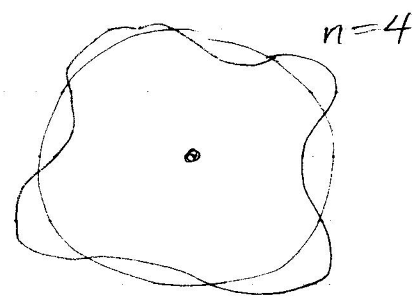
simple quantization condition!

de Broglie's interpretation (1924)

electron of momentum p has wavelength $\lambda = \frac{h}{p}$

standing wave: $2\pi r = n\lambda$

$$= n \frac{h}{m_e v}$$



$$\Longrightarrow m_e v r = n \frac{h}{2\pi}$$