Hidden Variables for Spin-$\frac{1}{2}$

Spin-$\frac{1}{2}$ particle has two independent spin states

$|\uparrow\rangle$: spin up along z-axis

$|\downarrow\rangle$: spin down along z-axis

most general spin state:

$$|\hat{n}\rangle = \cos\frac{\theta}{2}\, e^{-i\phi/2}\, |\uparrow\rangle + \sin\frac{\theta}{2}\, e^{+i\phi/2}\, |\downarrow\rangle$$

spin up in direction of unit vector $\hat{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$

measurement of $S_z = \vec{S}\cdot\hat{z}$

gives $+\frac{1}{2}\hbar$ with probability $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} = \frac{1+\hat{n}\cdot\hat{z}}{2}$

$-\frac{1}{2}\hbar$ " $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} = \frac{1+\hat{n}\cdot\hat{z}}{2}$

most general spin measurement: $\vec{S}\cdot\hat{m}$, where $\hat{m}$ is unit vector

gives $+\frac{1}{2}\hbar$ with probability $\frac{1+\hat{n}\cdot\hat{m}}{2}$

$-\frac{1}{2}\hbar$ " $\frac{1-\hat{n}\cdot\hat{m}}{2}$

According to Quantum Mechanics,
measurement is inherently probabilistic.

The Hidden Variable proposal is that measurements only appear to be probabilistic because there are additional hidden variables that have not been taken into account. The outcomes of all possible measurements are completely determined by the variables described by quantum mechanics and the hidden variables.

A spin-$\frac{1}{2}$ particle in the quantum state $|\hat{n}\rangle$ is actually in a classical state $(\hat{n}, h)$ specified by $\hat{n}$ and some hidden variables h.

A measurement of $\vec{S}\cdot\hat{m}$ in the state $|\hat{n}\rangle$ gives a result $A(\hat{n}, h; \hat{m})\cdot\frac{1}{2}\hbar$, that is uniquely determined by $\hat{n}$, $\hat{m}$, and the hidden variables.

In order to agree with quantum mechanics, the function $A(\hat{n}, h, \hat{m})$ must have the following properties:

- its only possible values are $+1$ and -1

- if averaged over the hidden variables, it must be $+1$ with probability $\frac{1+\hat{n}\cdot\hat{m}}{2}$
  -1 " $\frac{1+\hat{n}\cdot\hat{m}}{2}$

The simplest model for the hidden variables of a spin-$\frac{1}{2}$ particle is that they are specified by a single unit vector $\hat{h}$.

A spin-$\frac{1}{2}$ particle in a quantum state $|\hat{n}\rangle$ is actually in a classical state $(\hat{n}, \hat{h})$ specified by the two unit vectors.

A measurement of $\vec{S}\cdot\hat{m}$ in that state gives a unique result $A(\hat{n},\hat{h};\hat{m})\frac{1}{2}\hbar$, where

$$A(\hat{n},\hat{h};\hat{m}) = \text{sign}\left((\hat{n}+\hat{h})\cdot\hat{m}\right)$$

$$= +1 \quad \text{if} \quad (\hat{n}+\hat{h})\cdot\hat{m} > 0$$

$$= -1 \quad\quad (\hat{n}+\hat{h})\cdot\hat{m} > 0$$

If the hidden variable $\hat{h}$ has a uniform probability distribution on the unit sphere, the probability distribution of $A(\hat{n},\hat{h};m)$ is

$+1$ with probability $\frac{1+\hat{n}\cdot\hat{m}}{2}$

-1 " $\frac{1-\hat{n}\cdot\hat{m}}{2}$

Thus this hidden variable model reproduces deterministically the probabilistic results of measurement in quantum mechanics.