Commutators and Measurement

commutator of $\hat{A}$ and $\hat{B}$:

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Theorem: If $[\hat{A}, \hat{B}] = 0$,
$\hat{A}$ and $\hat{B}$ have simultaneous eigenstates

If a is an eigenvalue of $\hat{A}$
and b " $\hat{B}$,
there is a common eigenstate ψ:

$$\hat{A}\psi = a\psi$$
$$\hat{B}\psi = b\psi$$

Consequence for measurements:

If $[\hat{A}, \hat{B}] = 0$, measurements of $\hat{A}$ and $\hat{B}$
do not interfere with eachother.

If $\hat{A}$ is measured
and then $\hat{B}$ is (immediately) measured
and then $\hat{A}$ is (immediately) measured again,
the value for $\hat{A}$ will be the same as before.

If the measurement of $\hat{A}$ give the value a,
it collapses the wavefunction
to an eigenstate of $\hat{A}$ with eigenvalue a.

If the subsequent measurement of $\hat{B}$ give the value b,
it further collapse the wavefunction
to the simultaneous eigenstate of $\hat{A}$ and $\hat{B}$
with eigenvalues a and b.

A subsequent measurement of $\hat{A}$ will again give a
and not change the wavefunction

A subsequent measurement of $\hat{B}$ will again give b
and not change the wavefunction

Example Harmonic oscillator

Hamiltonian: $\hat{H} = \frac{1}{2m}\left(\frac{d}{dx}\right)^2 + \frac{1}{2}m\omega^2 x^2$

eigenvalues: $E_n = (n+\frac{1}{2})\hbar\omega \quad n = 0, 1, 2, \ldots$

eigenstates: $\psi_n(x) = H_n(x)\, e^{-m\omega x^2/2\hbar}$

where $H_n(x)$ is a polynomial of degree n
even polynomial if n is even
odd polynomial if n is odd

parity $\hat{\pi}$: $\hat{\pi}\psi(x) = \psi(-x)$

eigenvalues: $+1$ or -1

eigenfunctions

for $+1$: any even function $\psi(-x) = +\psi(x)$

for -1 : odd " $\psi(-x) = -\psi(x)$

$$[\hat{\pi}, \hat{H}] = 0$$

A measurement of parity will give

either $+1$ and collapse wavefunction to an even function

or -1 and " odd

If it gives -1,

subsequent measurement of energy

will give $(n+\frac{1}{2})\hbar\omega$ for some odd integer $n = 1$ or 3 or 5 or..

Subsequent measurement of parity

will continue to give -1

Subsequent measurement of energy

will continue to give the same value $(n+\frac{1}{2})\hbar\omega$.

Example Free particle

momentum operator $\hat{P}$

eigenvalues: any real number p

eigenfunction: $e^{ipx/\hbar}$

Hamiltonian operator: $\hat{H} = \frac{1}{2m}\hat{P}^2$

eigenvalues: any positive energy E

eigenfunctions: $A e^{i\sqrt{2mE}\,x/\hbar} + B e^{-i\sqrt{2mE}\,x/\hbar}$

where A and B are complex constants

If measurement of energy gives E,
wavefunction collapses to

$$A e^{i\sqrt{2mE}\,x/\hbar} + B e^{-i\sqrt{2mE}\,x/\hbar} \quad \text{for some } A, B$$

Subsequent measurement of momentum
will give either $+\sqrt{2mE}$ or $-\sqrt{2mE}$

If measurement gives $-\sqrt{2mE}$
wavefunction collapses further to $e^{i\sqrt{2mE}/\hbar}$

Subsequent measurement of energy and momentum
will continue to give E and $+\sqrt{2mE}$
and will not change the wavefunction