Black-Body Radiation

matter at temperature T
radiates electromagnetic waves

black-body: absorbs all incident radiation
reflects none
emits smooth spectrum

Stefan's Law 1879

power P radiated per area A
depends only on temperature T

$$P = \sigma A T^4$$

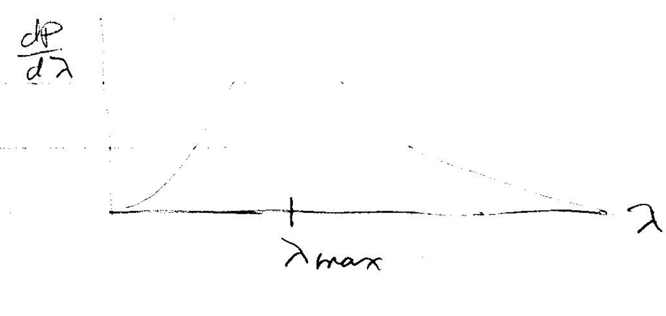
$$\sigma \approx 6 \times 10^{-8} \frac{J}{s \cdot m^2 \cdot K^4}$$ universal constant

Wien's Law 1893

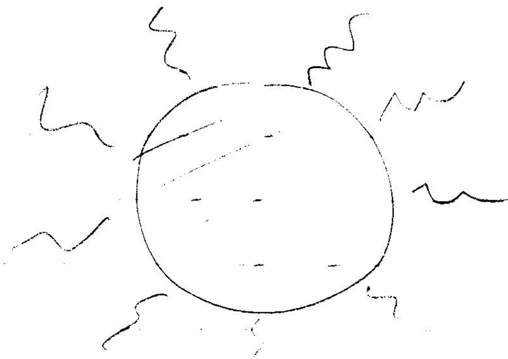
blackbody spectrum has a maximum
at a wavelength λ_{max} that depends only on T

$$\lambda_{max} = \frac{W}{T}$$

$$W \approx 3 \cdot 10^{-3} \ m \cdot K$$

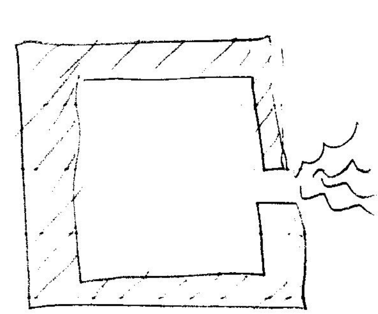


Radiation from black body
at temperature T



is identical to

Radiation emitted by hole
in cavity whose walls
are at temperature T



$$P_{hole} = \sigma A_{hole} T^4$$

Travelling EM waves emitted from hole
are related to standing EM waves in cavity

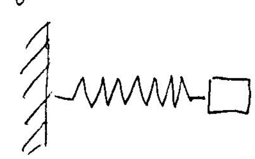
$$P_{hole} = A_{hole} \left(\tfrac{1}{4} c\right) \rho_{cavity}$$

angle average of velocity of light

ρ_{cavity} = energy density of standing EM waves

ρ_{cavity} can be expressed
as integral over frequencies of normal modes

$$\rho_{cavity} = \underbrace{\int_0^\infty \frac{8\pi \nu^2 d\nu}{c^3}} \bar{E}(\nu)$$



sum over normal modes
with frequency ν
per volume

average energy
of normal mode
of frequency ν.

frequency spectrum of black-body radiation

$$dP = A\left(\frac{1}{4}c\right)\frac{8\pi}{c^3}\,\bar{E}(\nu)\,\nu^2\,d\nu$$

$\bar{E}(\nu)$ = average energy
in standing EM wave with frequency ν
in equilibrium at temperature T

Classical thermodynamics

harmonic oscillator
in equilibrium at temperature T

has energy distribution $\propto e^{-E/kT}$

average energy

$$\bar{E} = \frac{\int_0^\infty dE\, E\, e^{-E/kT}}{\int_0^\infty dE\, e^{-E/kT}}$$

$$= \frac{(kT)^2}{kT} = kT$$

predicted black-body spectrum

$$dP = A\,\frac{2\pi}{c^2}\,kT\,\nu^2\,d\nu$$

Planck: fitting formula for spectrum

$$dP = A \cdot \frac{2\pi}{c^2} kT \frac{h\nu/kT}{e^{h\nu/kT}-1} \nu^2 d\nu$$

where h is a universal constant

$$h \approx 7 \times 10^{-34} \text{ J} \cdot \text{s}$$

Suppose energy of normal mode of frequency ν can only be integer multiple of $h\nu$

$$E = 0 \text{ or } h\nu \text{ or } 2h\nu \text{ or } \ldots$$

$$E = nhf, \quad n = 0, 1, 2, \ldots$$

average energy?

$$\bar{E}(\nu) = \frac{\sum_{n=0}^{\infty} nh\nu \, e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}}$$

$$= \frac{kT^2 \frac{d}{dT} \frac{-1}{1-e^{-h\nu/kT}}}{\frac{1}{1-e^{-h\nu/kT}}}$$

$$= \frac{h\nu}{e^{h\nu/kT}-1}$$

integrate Planck's formula over ν
to get total power

$$P = A \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

change to dimensionless variable

$$x = \frac{h\nu}{kT} \qquad dx = \frac{h}{kT} d\nu$$

$$\nu^3 d\nu = \left(\frac{kT}{h}\right)^4 x^3 dx$$

$$P = A \frac{2\pi h}{c^2} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

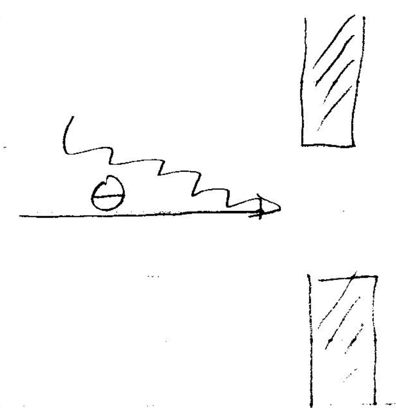
$$= A \sigma T^4$$

Stefan's constant

$$\sigma = \frac{2\pi k^4}{c^2 h^3} \underbrace{\int_0^\infty \frac{x^3 dx}{e^x - 1}}_{6.49}$$

expressed in terms of fundamental constants
c, k, h

Light waves near hole in cavity



velocity component of light perpendicular to hole: $v_x = c\cos\theta$

$0 < \theta < \frac{\pi}{2}$: $v_x > 0$ (approaching hole)
$\frac{\pi}{2} < \theta < \pi$: $v_x < 0$

angle average of v_x for light approaching hole

$$\frac{\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta\, (c\cos\theta) + \int_{\frac{\pi}{2}}^{\pi} \sin\theta\, d\theta \cdot 0}{\int_0^{\pi} \sin\theta\, d\theta \cdot 1}$$

$$= \frac{c\int_0^{\pi/2} d\theta\, \sin\theta\cos\theta}{\int_0^{\pi} d\theta\, \sin\theta} = c\,\frac{\int_0^{\pi/2} d(\cos^2\theta/2)}{\int_0^{\pi} d(\cos\theta)}$$

$$= c\,\frac{1/2}{2} = \frac{c}{4}$$

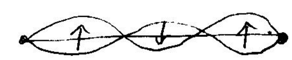
Wave equation in 1 dimension
for waves with speed c

$$\left[\left(\frac{\partial}{\partial t}\right)^2 - c^2\left(\frac{\partial}{\partial x}\right)^2\right] f = 0$$

travelling waves: 

$$f(x,t) = \sin(kx - \omega t) \qquad \omega = kc$$

standing waves in interval $0 < x < L$
that vanish at endpoints: $f = 0$ at $x = 0$
at $x = L$

$$f(x,t) = \sin(kx)\sin(\omega t)$$ 

$$k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \qquad \omega = kc = \frac{n\pi c}{L}$$

discrete frequencies: $\nu_n = \frac{\omega_n}{2\pi} = \frac{c}{2L} n$

difference between adjacent frequencies

$$\Delta\nu_n = \frac{c}{2L}\Delta n, \quad \Delta n = 1$$

Sum over normal modes:

$$\sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \Delta n = \sum_{n=1}^{\infty} \frac{2L}{c}\Delta\nu_n \rightarrow \frac{2L}{c}\int_0^{\infty} d\nu$$

Standing EM waves in 3 dimensions
in cubic cavity of length L

$$\vec{E}(x,y,z,t) = \vec{E}_0 \sin(k_1 x)\sin(k_2 y)\sin(k_3 z)\sin(\omega t)$$

2 polarizations $\Rightarrow$ 2 solutions for $\vec{E}_0$

wave equation: $\omega^2 = c^2(k_1^2 + k_2^2 + k_3^2)$

boundary condition: $k_1 = \frac{n_1 \pi}{L}$ $\quad n_1 = 1,2,3,\ldots$
$k_2 = n_2\pi/L$ $\quad n_2 = 1,2,3,\ldots$
$k_3 = n_3\pi/L$ $\quad n_3 = 1,2,3,\ldots$

Sum over normal modes

$$2\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty} \rightarrow 2\,\frac{L}{\pi}\int_0^\infty dk_1\,\frac{L}{\pi}\int_0^\infty dk_2\,\frac{L}{\pi}\int_0^\infty dk_3$$

$$= 2\frac{L^3}{\pi^3}\int_{k_1>0,\,k_2>0,\,k_3>0} d^3k$$

$$= 2\frac{L^3}{\pi^3}\,\frac{1}{8}\,4\pi\int_0^\infty k^2\,dk \qquad k = \sqrt{k_1^2+k_2^2+k_3^2} = \frac{\omega}{c} = \frac{2\pi\nu}{c}$$

$$= 2\frac{L^3}{\pi^3}\,\frac{\pi}{2}\left(\frac{2\pi}{c}\right)^3\int_0^\infty \nu^2\,d\nu$$

$$= 2\,\frac{4\pi}{c^3}\,L^3\int_0^\infty \nu^2\,d\nu$$

$$= V\cdot\frac{8\pi}{c^3}\int_0^\infty \nu^2\,d\nu$$