3-Dimensional Harmonic Oscillator

$$V(x,y,z) = \frac{1}{2} m \omega^2 (x^2+y^2+z^2)$$

Schroedinger equation for definite energy E

$$-\frac{\hbar^2}{2m}\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 \right)\psi$$

$$+ \frac{1}{2} m\omega^2 (x^2+y^2+z^2)\,\psi = E\psi$$

Look for simple solutions
in which dependence in x, y, z is separated

$$\psi(x,y,z) = \psi_1(x)\,\psi_2(y)\,\psi_3(z)$$

$$\left(\frac{\partial}{\partial x}\right)^2 \psi(x,y,z) = \psi_1''(x)\,\psi_2(y)\,\psi_3(z)$$

$$-\frac{\hbar^2}{2m}\left[\psi_1''(x)\,\psi_2(y)\,\psi_3(z) + \psi_1(x)\,\psi_2''(y)\,\psi_3(z) + \psi_1(x)\,\psi_2(y)\,\psi_3''(z) \right]$$

$$+ \frac{1}{2} m\omega^2 (x^2+y^2+z^2)\,\psi_1(x)\,\psi_2(y)\,\psi_3(z) = E\,\psi_1(x)\,\psi_2(y)\,\psi_3(z)$$

Divide both sides by $\psi_1(x)\,\psi_2(y)\,\psi_3(z)$

$$-\frac{\hbar^2}{2m}\left(\frac{\psi_1''(x)}{\psi_1(x)} + \frac{\psi_2''(y)}{\psi_2(y)} + \frac{\psi_3''(z)}{\psi_3(z)} \right) + \frac{1}{2} m\omega^2 (x^2+y^2+z^2) = E$$

The dependence on x, y, and z can now be separated.

$$\left(-\frac{\hbar^2}{2m}\frac{\psi_1''(x)}{\psi_1(x)}+\frac{1}{2}m\omega^2x^2\right)+\left(-\frac{\hbar^2}{2m}\frac{\psi_2''(y)}{\psi_2(y)}+\frac{1}{2}m\omega^2y^2\right)$$

$$+\left(-\frac{\hbar^2}{2m}\frac{\psi_3''(z)}{\psi_3(z)}+\frac{1}{2}m\omega^2z^2\right)=E$$

A function of x only, a function of y only, and a function of z only can add up to a constant E only if each of the functions is separately constant. Call the constants E_1, E_2, and E_3.

$$-\frac{\hbar^2}{2m}\frac{\psi_1''(x)}{\psi_1(x)}+\frac{1}{2}m\omega^2x^2=E_1$$

$$-\frac{\hbar^2}{2m}\frac{\psi_2''(y)}{\psi_2(y)}+\frac{1}{2}m\omega^2y^2=E_2$$

$$-\frac{\hbar^2}{2m}\frac{\psi_3''(z)}{\psi_3(z)}+\frac{1}{2}m\omega^2z^2=E_3$$

The sum of the three constants must of course be E:

$$E_1+E_2+E_3=E$$

After multiplying by the wavefunction in the denominator, we get three separate 1-dimensional Schroedinger equations:

$$-\frac{\hbar^2}{2m}\psi_1''(x)+\frac{1}{2}m\omega^2x^2\psi_1(x)=E_1\psi_1(x)$$

$$-\frac{\hbar^2}{2m}\psi_2''(y)+\frac{1}{2}m\omega^2y^2\psi_2(y)=E_2\psi_2(y)$$

$$-\frac{\hbar^2}{2m}\psi_3''(z)+\frac{1}{2}m\omega^2z^2\psi_3(z)=E_3\psi_3(z)$$

Recall the solution to the 1-dimensional Schroedinger equation for the harmonic oscillator:

$$-\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x)$$

The energy E has discrete eigenvalues:

$$E_n = (n+\tfrac{1}{2})\hbar\omega, \quad n=0,1,2,\ldots$$

The corresponding wavefunctions have the form

$$\psi_n(x) = H_n(x)\, e^{-m\omega x^2/2\hbar}$$

where $H_n(x)$ is a polynomial of degree n.

We can now write down the solution to the 3-dimensional Schroedinger equation. The separation constants E_1, E_2, E_3 have discrete values labelled by quantum numbers n_1, n_2, n_3

$$E_1 = (n_1+\tfrac{1}{2})\hbar\omega \quad n_1=0,1,2,\ldots$$

$$E_2 = (n_2+\tfrac{1}{2})\hbar\omega \quad n_2=0,1,2,\ldots$$

$$E_3 = (n_3+\tfrac{1}{2})\hbar\omega \quad n_3=0,1,2,\ldots$$

The discrete energy eigenvalue are

$$E_{n_1, n_2, n_3} = \left(n_1 + n_2 + n_3 + \tfrac{3}{2}\right)\hbar\omega$$

The corresponding wavefunction are

$$\psi_{n_1, n_2, n_3}(x, y, z) = \left(H_{n_1}(x)\, e^{-m\omega x^2/2\hbar}\right)\left(H_{n_2}(y)\, e^{-m\omega^2 y^2/2\hbar}\right)$$

$$\times \left(H_{n_3}(z)\, e^{-m\omega^2 z^2/2\hbar}\right.$$

$$= H_{n_1}(x)\, H_{n_2}(x)\, H_{n_3}(x)\, e^{-m\omega(x^2+y^2+z^2)/2\hbar}$$

There is a unique lowest-energy state

$$E = \tfrac{3}{2}\hbar\omega \qquad (n_1, n_2, n_3) = (0,0,0)$$

The first excited state has degeneracy 3:

$$E = \tfrac{5}{2}\hbar\omega \qquad (n_1, n_2, n_3) = (1,0,0)$$
$$= (0,1,0)$$
$$= (0,0,1)$$