Schwartz Chapter 22: Problems 2*, 3*

Problem 2*

Determine the terms of order $1/M^2$ and $1/M^4$ in the effective Lagrangian for the 4-Fermi effective field theory that provides a low-energy approximation to the fundamental field theory defined by the Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{free}} &= -\tfrac{1}{4}W_{\mu\nu}W^{\mu\nu} + \tfrac{1}{2}M^2 W_\mu W^\mu + \bar{\psi}(i\partial\!\!\!/ - m)\psi, \\ \mathcal{L}_{\text{int}} &= gW_\mu \bar{\psi}\gamma^\mu \psi, \end{aligned}$$

where $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$. Include the fermion mass term and assume that $m \ll M$.

A. Write down the tree-level matrix element $\mathcal{M}$ for $\psi\bar{\psi} \to \psi\bar{\psi}$, as in Eq. (22.15) but also including the t-channel diagram.

B. Consider low-energy scattering with center-of-mass energy $E_{\text{cm}} \ll M$. Expand $\mathcal{M}$ to leading order in p/M, where p is any external momentum, as in Eq. (22.16) but also including the t-channel diagram.

C. Identify a 4-fermion term in the effective Lagrangian that reproduces the matrix element in part B. Write down the Feynman rule for the $\psi\psi\bar{\psi}\bar{\psi}$ vertex, making the spinor indices on the fermion lines explicit.

D. Consider low-energy scattering with center-of-mass energy $E_{\text{cm}} \ll M$. Expand the matrix element $\mathcal{M}$ in part A to next-to-leading order in p/M. Show that the $q_\mu q_\nu$ term from the propagator of a vector meson of momentum q does not contribute to $\mathcal{M}$.

E. Identify a 4-fermion term in the effective Lagrangian that reproduces the next-to-leading order term in the matrix element in part D. Write down the Feynman rule for the $\psi\psi\bar{\psi}\bar{\psi}$ vertex, making the spinor indices on the fermion lines explicit.

Problem 3*

A. Use the Pauli algebra to derive the trace identities

$$\begin{aligned}\mathrm{Tr}[\sigma^a\sigma^b] &= 2\delta^{ab},\\ \mathrm{Tr}[\sigma^a\sigma^b\sigma^c\sigma^d] &= 2\left(\delta^{ab}\delta^{cd}-\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right).\end{aligned}$$

Use the Pauli algebra to show that $(\pi^a\sigma^a)^2 = \pi^a\pi^a\,\mathbb{1}$.
The last expression for the chiral field $U(x)$ in Eq. (22.17) treats the pion fields π^1, π^2, and π^3 symmetrically. Show that the expansion for $U(x)$ to 4th order in the pion fields is

$$U(x) = \mathbb{1} + \frac{i}{F_\pi}\pi^a\,\sigma^a - \frac{1}{2F_\pi^2}\pi^a\pi^a\,\mathbb{1} - \frac{i}{6F_\pi^3}\pi^b\pi^b\pi^a\,\sigma^a + \frac{1}{24F_\pi^4}\pi^a\pi^a\pi^b\pi^b\,\mathbb{1} + \ldots.$$

B. The chiral Lagrangian $\mathcal{L}_\chi$ in Eq. (22.18) in the absence of electromagnetic fields is

$$\mathcal{L}_\chi = \frac{F_\pi^2}{4}\,\mathrm{Tr}[\partial_\mu U\partial_\mu U^\dagger].$$

Expand $\mathcal{L}_\chi$ to 4th order in the pion fields.
From the 2nd order terms, deduce that the propagator for a pion of momentum k and isospin indices a and b is

$$\frac{i\delta^{ab}}{k^2+i\epsilon}.$$

From the 4th order terms, deduce that the 4-pion vertex with incoming momenta k_1, k_2, k_3, k_4 and with isospin indices a, b, c, d is

$$\begin{aligned}\frac{i}{3F_\pi^2}\big[&(k_1.k_2+k_3.k_4)(\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}-2\delta^{ab}\delta^{cd})\\ &+(k_1.k_3+k_2.k_4)(\delta^{ab}\delta^{cd}+\delta^{ad}\delta^{bc}-2\delta^{ac}\delta^{bd})\\ &+(k_1.k_4+k_2.k_3)(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}-2\delta^{ad}\delta^{bc})\big].\end{aligned}$$

C. Write down the tree-level matrix element for $2\to2$ pion scattering through the 4-pion vertex for the process $(p_1,a)\ (p_2,b)\longrightarrow(p_3,c)\ (p_4,d)$. Express the matrix element in terms of Mandelstam variables, and simplify it if possible using the identity $s+t+u=0$.

D. The term in the Lagrangian that breaks the $SU(2)_L \times SU(2)_R$ chiral symmetry with the same pattern as the quark masses in QCD is

$$\mathcal{L}_{\rm mass} = B\left(\mathrm{Tr}[UM^\dagger] + \mathrm{Tr}[MU^\dagger]\right),$$

where $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$ is the diagonal matrix of up and down quark masses.
Expand $\mathcal{L}_{\rm mass}$ to 4th order in the pion fields. (It may help to express M as a linear combination of $\mathbb{1}$ and σ^3.)
By expressing the 2nd order terms in $\mathcal{L}_{\rm mass}$ in the form $-\frac{1}{2}m_\pi^2\pi^a\pi^a$, deduce the relation between the coefficient B and the pion mass m_π.
From the 2nd order terms in $\mathcal{L}_\chi + \mathcal{L}_{\rm mass}$, deduce the pion propagator.
From the 4th order terms in $\mathcal{L}_\chi + \mathcal{L}_{\rm mass}$, deduce the 4-pion vertex.

E. Write down the tree-level matrix element for $2 \to 2$ pion scattering through the 4-pion vertex for the process $(p_1, a)\ (p_2, b) \longrightarrow (p_3, c)\ (p_4, d)$. Express the matrix element in terms of Mandelstam variables, and simplify it if possible using the identity $s + t + u = 4m_\pi^2$.