Schwartz Chapter 20: Problem 6*

Problem 6*

The matrix element for a high-energy reaction with an incoming electron of momentum P can be expressed in the form

$$\mathcal{M}_0 = \bar{\Gamma}\, u(P),$$

where $\bar{\Gamma}$ has a Dirac index contracted with that of $u(P)$.

(A) Express $|\mathcal{M}_0|^2$ as a trace involving the factor $\Gamma\bar{\Gamma}$.

(B) If the electron energy is so large that its mass can be neglected, the sum of the electron spinor factor over its spin states is

$$\sum_{\text{spins}} u(P)\bar{u}(P) = \not{P}.$$

Use this to simplify the expression for $\sum_{\text{spins}} |\mathcal{M}_0|^2$.

The same reaction can occur with the radiation of a photon of momentum q. The contribution to the matrix element from the diagram in which the photon is radiated from the line of the incoming electron can be expressed in the form

$$\mathcal{M}_1 = \bar{\Gamma}\, \frac{i}{\not{P} - \not{q}}\, (-ie\gamma^\alpha)\, u(P)\, \varepsilon^*_\alpha(q).$$

(We have assumed that the electron energy is so large that m_e can be ignored in the electron propagator.) If the photon is nearly collinear to the incoming electron, the other diagrams can be neglected. The factor $\Gamma\bar{\Gamma}$ is essentially the same as in $\mathcal{M}_0$ except that P in the momentum-conservation condition is replaced by $P - q$.

(C) Express $|\mathcal{M}_1|^2$ as a trace involving the factor $\Gamma\bar{\Gamma}$ divided by the denominator $(2P.q)^2$.

(D) The sum of the photon wavefunction factor over its physical spin states is

$$\sum_{\text{spins}} \varepsilon^*_\alpha(q)\,\varepsilon_\beta(q) = -g_{\alpha\beta} + \frac{q_\alpha \bar{q}_\beta + \bar{q}_\alpha q_\beta}{q.\bar{q}}.$$

Use this to obtain an expression for $\sum_{\text{spins}} |\mathcal{M}_1|^2$, where the sum is over the spins of the incoming electron and the physical spins of the radiated photon.

(E) Use Dirac algebra to simplify the trace. Reduce the expression inside the trace to the product of $\Gamma\bar{\Gamma}$ and a linear combination of $\not{P}$, $\not{q}$, and $\not{\bar{q}}$. (You should be able to cancel one of the factors of $2P.q$ in the denominator.)

(F) The collinear approximation for the photon momentum is $q \approx zP$, where the longitudinal momentum z is in the range $0 \leq z \leq 1$. Use this to simplify $2P.q \sum_{\text{spins}} |\mathcal{M}_1|^2$, reducing it to $\mathrm{Tr}(\Gamma\bar{\Gamma}\not{P})$ multiplied by a function of z.

(G) Show that the phase space integration measure for the photon can be expressed as

$$\int \frac{d^3q}{(2\pi)^3 2q_0} = \frac{1}{4\pi}\int_0^1 \frac{dz}{z} \int \frac{d^2q_\perp}{(2\pi)^2}, \qquad z = \frac{q_0 + q_z}{P_0 + P_z},$$

where q_z is the component of $\vec{q}$ along the direction of $\vec{P}$ and $\vec{q}_\perp$ is its perpendicular component.

(H) Show that if the electron mass is neglected, the denominator factor can be expressed as

$$\frac{1}{2P.q} = \frac{z}{q_\perp^2}.$$

(I) If the collision after the radiation of the photon involves a large momentum transfer Q satisfying $Q \gg m_e$, the greatest sensitivity to $\vec{q}_\perp$ comes from the denominator factor $1/(2P.q)$. Show that the integral of that factor over the region $m_e < |\vec{q}_\perp| < Q$ of the photon phase space can be approximated as

$$\int \frac{d^3q}{(2\pi)^3 2q_0}\frac{1}{2P.q} \approx \frac{1}{16\pi^2}\log\frac{Q^2}{m_e^2}\int dz.$$

(J) The integral of $\sum_{\text{spins}} |\mathcal{M}_1|^2$ over the photon phase space can be approximated as

$$\int \frac{d^3q}{(2\pi)^3 2q_0} \sum_{\text{spins}} |\mathcal{M}_1|^2 \approx \int_0^1 dz\, f(z)\, \mathrm{Tr}\big(\Gamma\bar{\Gamma}\not{P}\big).$$

Determine the function $f(z)$.

(K) For the collision of an electron of momentum $(1-z)P$, the square of the matrix element summed over spins is

$$\sum_{\text{spins}} |\mathcal{M}_0|^2 = \mathrm{Tr}\big(\Gamma\bar{\Gamma}\,(1-z)\not{P}\big).$$

It differs from that for an electron of momentum P by the factor $1-z$. Determine how the flux factor for the collision of an electron of momentum $(1-z)P$ differs from that for an electron of momentum P.

(L) Show that the differential cross section for scattering of a high-energy electron of momentum P with the radiation of a collinear photon can be expressed in terms of the differential cross section for scattering of an electron of momentum $(1-z)P$ without radiation:

$$d\sigma_1\big[e^-(P)\big] = \int_0^1 dz\, f(z)\, d\sigma_0\big[e^-((1-z)P)\big]$$

Thus $f(z)$ can be interpreted as the probability that radiation produces a collinear electron with momentum $(1-z)P$.