Schwartz Chapter 19: Problem 1*, 2*

Problem 1*

(a) The photon self-energy function at order e^2 (Problem 16.1) with dimensional regularization to regularize the ultraviolet divergence is

$$\Pi(q^2) = \delta_3 - \frac{e^2}{8\pi^2}\int_0^1 dx\,(1-x)(1-2x)\left(\frac{2}{4-d} - \log\frac{m^2 - x(1-x)q^2 - i\epsilon}{\bar{\mu}^2}\right),$$

where $\bar{\mu}^2 = 4\pi e^{-\gamma}\mu^2$. (The denominator of the complete photon propagator is $q^2[1+\Pi(q^2)]$.) Specify the on-shell renormalization condition and determine the counterterm δ_3 in the limit $d \to 4$.

(b) The scalar self-energy function at order e^2 (Problem 18.1) with dimensional regularization to regularize the ultraviolet divergences and a photon mass m_γ to regularize an infrared divergence is

$$\begin{aligned}\Sigma(p^2) &= \delta_2(p^2-m^2) + \delta_m m^2 \\ &\quad + \frac{e^2}{4\pi^2}\int_0^1 dx\left(\left[2xm^2 + (4-6x+3x^2)p^2\right] + \frac{4-d}{2}\left[xm^2 - x(1-x)p^2\right]\right) \\ &\qquad \times\left(\frac{2}{4-d} - \log\frac{xm^2 - x(1-x)p^2 + (1-x)m_\gamma^2 - i\epsilon}{\bar{\mu}^2}\right).\end{aligned}$$

(The denominator of the complete scalar propagator is $p^2 - \Sigma(p^2)$.) Specify the on-shell renormalization conditions and determine the counterterms δ_m and δ_2 in the limits $d \to 4$ and $m_\gamma \to 0$.

(c) The 1PI photon-scalar vertex can be expressed as $-ieG^\mu(p', p)$, where p and p' are the incoming and outgoing scalar momenta. The vertex at order e^2 is $G_0^\mu(p', p) = (p+p')^\mu$. The order-$e^2$ correction $G_1^\mu(p', p)$ to the vertex is the sum of a counterterm $\delta_1(p+p')^\mu$ and 3 one-loop diagrams, one with 3 internal lines and two with 2 internal lines. The on-shell renormalization condition is that there are no corrections to the vertex with the scalars on their mass shells in the limit of zero momentum transfer:

$$G^\mu(p', p)\Big|_{p^2=p'^2=m^2,\, q^2=0} = (p+p')^\mu.$$

Calculate the 1-loop diagrams using dimensional regularization to regularize the ultraviolet divergence and a photon mass m_γ to regularize the infrared divergence. To evaluate each diagram,
1. Draw the diagram, labeling the momenta and Lorentz indices.
2. Write down the mathematical expression using the Feynman rules, but with a photon mass in the virtual photon propagator.
3. Set the external scalars on their mass shells ($p^2 = p'^2 = m^2$) and take the limit of zero momentum transfer ($q^2 = 0$).
4. Combine the denominators using Feynman parameters.
5. Shift the loop momentum k to make the denominator an even function of k.
6. Average the numerator over "angles" in d dimensions.
7. Integrate over the loop momentum.
8. Expand to 0th order in $d - 4$.
9. Evaluate the Feynman parameter integrals in the limit $m_\gamma \to 0$.
Add the 3 diagrams and determine the counterterm δ_1.

Problem 2*

The Ward-Takahashi identity in coordinate space (Problem 14.5) is

$$\frac{\partial}{\partial x^\mu}\big\langle J^\mu(x)\phi(y)\phi^*(z)\big\rangle = -\delta^4(x-y)\big\langle\phi(x)\phi^*(z)\big\rangle + \delta^4(x-z)\big\langle\phi(y)\phi^*(x)\big\rangle.$$

(a) Derive the Ward-Takahashi identity in momentum space.
First Fourier transform the coordinates x, y, and z to momenta q, $-p'$, and p, factoring out an overall delta function $\delta^4(p'-p-q)$. The resulting identity involves the connected $J^\mu\phi\phi^*$ Green function $G^\mu_{\rm conn}(p',p)$ and the scalar propagators $D(p^2)$ and $D(p'^2)$. Then multiply by inverse propagators to obtain an identity for the 1PI $J^\mu\phi\phi^*$ Green function $G^\mu(p',p)$:

$$G^\mu(p',p) = D(p'^2)^{-1}\, G^\mu_{\rm conn}(p',p)\, D(p^2)^{-1}.$$

(b) Describe how Z_1 and Z_2 are determined by the on-shell renormalization conditions on $G^\mu(p',p)$ and on $D(p^2)$. Then use the Ward-Takahashi identity to derive the identity $Z_1 = Z_2$.