Schwartz Chapter 16: Problems 1*, 3*, 4*

Problem 1*

Calculate the vacuum polarization graph in spinor QED only. The relevant Ward identity is $p_\mu \Pi^{\mu\nu}(p) = 0$. Calculate the $g^{\mu\nu}$ terms as well as the $p^\mu p^\nu$ terms. The calculation involves the following steps:
1. Draw the diagram, labelling the momenta and Lorentz indices.
2. Write down the mathematical expression using the Feynman rules.
3. Combine the scalar denominators using a Feynman parameter.
4. Shift the loop momentum k so the denominator is a function of k^2 only.
5. Average the numerator over angles in d space-time dimensions by using the substitution $k^\mu k^\nu \to k^2 g^{\mu\nu}/d$.
6. Calculate the traces of the Dirac matrices.
7. Evaluate the momentum integral in d space-time dimensions.
Your final result should be left as an integral over the Feynman parameter.

Problem 3*

(0) The calculation of $\mathcal{M}(s,t)$ involves the following steps:
1. Draw the tree-level s-channel diagram and the one-loop s-channel diagram, labelling the momenta and the Lorentz indices.
2. Write down the mathematical expression for $i\mathcal{M}$ using the Feynman rules. (For the τ lepton loop inserted into a virtual photon line of momentum q, use the Feynman rule $i\Pi^{\mu\nu}(q) = -i\Pi(q^2)(q^2 g^{\mu\nu} - q^\mu q^\nu)$.)
3. Use Lorentz algebra to express $\mathcal{M}$ in terms of scalar products and $\Pi(q^2)$.
4. Renormalize the self-energy function $\Pi(q^2)$ by subtracting its value at $q^2 = 0$: $\Pi(q^2) = e^2[\Pi_2(q^2) - \Pi_2(0)]$, where $\Pi_2(q^2)$ is Schwartz's result in Eq. (16.49) with the appropriate $i\epsilon$ prescription inserted.
5. Express your final result for $\mathcal{M}$ in terms of the Mandelstam variables s and t, leaving the term of order e^2 as an integral over the Feynman parameter.

(a) Plot $|\mathcal{M}(s,t=0)|^2$ as a function of $\sqrt{s}$ on a log scale from 100 MeV to 10 GeV. Identify the energy $\sqrt{s_0}$ where there is a kink.

(b) The imaginary part of $\Pi(q^2)$ comes from the $i\epsilon$ prescription inherited from the tau lepton propagators. To plot the real and imaginary parts of $\mathcal{M}$ numerically, you can just take ϵ to be a tiny positive number. To calculate $\text{Im}\mathcal{M}$ explicitly, you need to use the limit $\epsilon \to 0^+$ to determine the appropriate branch of the logarithm. Evaluate the simple Feynman parameter integral in the imaginary part analytically.

(1) Calculate the cross section for $\pi^+\pi^- \to \tau^+\tau^-$ through the following steps:
1. Draw the tree-level diagram, labeling the momenta and Lorentz indices.
2. Write down the mathematical expression for $i\mathcal{M}$ using the Feynman rules.
3. Express $\mathcal{M}\mathcal{M}^*$ in terms of spinor products.
4. Express $\sum_{\text{spins}} |\mathcal{M}|^2$ in terms of Dirac traces.
5. Evaluate the traces and reduce $\sum |\mathcal{M}|^2$ to scalar products.
6. Express $\sum |\mathcal{M}|^2$ in terms of the Mandelstam variables s and t.
7. Express t in terms of the scattering angle θ in the CM frame.
8. Reduce the differential phase space $d\Pi$ to a differential of $\cos\theta$.
9. Integrate over θ to obtain the rate $\int \sum |\mathcal{M}|^2 d\Pi$.
10. Multiply by the flux factor to get the cross section $\sigma(s)$.

(c) Determine the multiplicative relation between $\text{Im}\mathcal{M}(s, t = 0)$ from part (0) and the cross section $\sigma(s)$ for $\pi^+\pi^- \to \tau^+\tau^-$ from part (1).

<u>Problem 4</u> (hints)

The rate at which the QED coupling constant runs increases at the threshold for each charged elementary particle. The thresholds for the electron, muon, and τ lepton are just their masses. A reasonable choice for the threshold for a top quark is also its mass. A reasonable choice for the threshold of a lighter quark is the mass of the lightest spin-1 meson containing that quark: $\rho(770)$ for u and d, $K^*(892)$ for s, $D^*(2010)$ for c, and $B^*(5330)$ for b. Determine $1/\alpha(\mu)$ numerically at each threshold. Use the value $\alpha(m_t)$ at the top-quark mass to determine the value of the Landau pole in the absence of weak interactions.