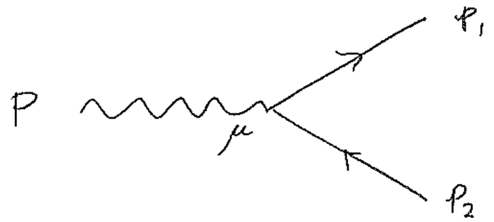
Soft Photons in Z^0 Decay

The Feynman rule for the photon-muon vertex is $-ie\gamma^\mu$.
The Feynman rule for the Z^0-muon vertex is $i(g_V - g_A\gamma_5)\gamma^\mu$.
The wavefunction factor for an external Z^0 line of momentum P is $\varepsilon_\mu(P)$.

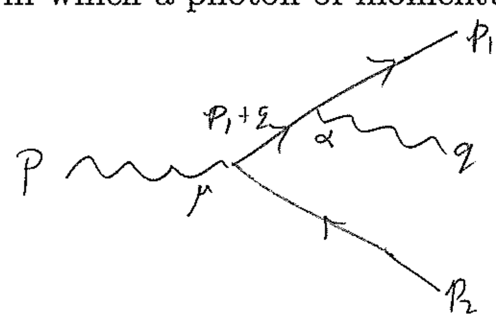
A. Draw the tree-level diagram for the decay $Z^0(P) \to \mu^-(p_1)\mu^+(p_2)$.

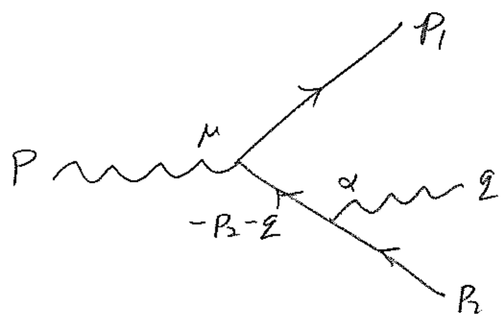


B. Write down the matrix element $\mathcal{M}_0$ for the decay $Z^0 \to \mu^-\mu^+$.

$$i\mathcal{M}_0 = \bar{u}(p_1)\, i(g_V - g_A\gamma_5)\gamma^\mu v(p_2)\, \varepsilon_\mu(P)$$

C. Draw the two tree-level diagrams for the decay $Z^0(P) \to \mu^-(p_1)\mu^+(p_2)\gamma(q)$ in which a photon of momentum q is radiated.





D. Write down the matrix element $\mathcal{M}_1$ for the decay $Z^0 \to \mu^-\mu^+\gamma$.

$$i\mathcal{M}_1 = \varepsilon_\mu(P)\Big[\bar{u}(p_1)(-ie\gamma^\alpha)\frac{i}{\not p_1 + \not q - m + i\epsilon}(-ie\gamma^\mu)v(p_2)$$

$$+ \bar{u}(p_1)(-ie\gamma^\mu)\frac{i}{-\not p_2 - \not q - m + i\epsilon}(-ie\gamma^\alpha)v(p_2)\Big]\varepsilon^*_\alpha(q)$$

The matrix element $\mathcal{M}_1$ for the decay $Z^0(P) \to \mu^-(p_1)\mu^+(p_2)\gamma(q)$ is

$$\mathcal{M}_1 = e\,\varepsilon_\mu(P)\,\varepsilon^*_\alpha(q)\,\bar{u}(p_1)\left(\gamma^\alpha \frac{\not p_1 + \not q + m}{(p_1+q)^2 - m^2}(g_V - g_A\gamma_5)\gamma^\mu \right.$$
$$\left. +(g_V - g_A\gamma_5)\gamma^\mu \frac{-\not p_2 - \not q + m}{(p_2+q)^2 - m^2}\gamma^\alpha\right)v(p_2).$$

The spinors satisfy $\bar{u}(p_1)(\not p_1 - m) = 0$ and $(\not p_2 + m)v(p_2) = 0$.
The photon polarization vector satisfies $q^\alpha \varepsilon_\alpha(q) = 0$.

The term in $\mathcal{M}_1$ from radiation of γ from the μ^- line has the factor

$$\varepsilon^*_\alpha(q)\,\bar{u}(p_1)\gamma^\alpha \frac{\not p_1 + \not q + m}{(p_1+q)^2 - m^2}.$$

E. Simplify the denominator using the facts that the μ^- and γ are on shell.

$$(p_1+q)^2 - m^2 = p_1^2 + 2p\cdot q + q^2 - m^2 = m^2 + 2p\cdot q + 0 - m^2 = 2p\cdot q$$

F. Use Dirac algebra to eliminate $\not p_1$ from the numerator.

$$\bar{u}(p_1)\,\gamma^\alpha(\not p_1 + \not q + m) = \bar{u}(p_1)\left[2p_1^\alpha - \not p_1\gamma^\alpha + \gamma^\alpha(\not q + m)\right]$$
$$= \bar{u}(p_1)\left[2p_1^\alpha - \cancel{m\gamma^\alpha} + \gamma^\alpha(\not q + \cancel{m})\right] = \bar{u}(p_1)(2p_1^\alpha + \gamma^\alpha\not q)$$

G. In the soft-photon limit, express the term above as a simple scalar factor multiplying $\bar{u}(p_1)$.

$$\varepsilon^*_\alpha(q)\,\bar{u}(p_1)\,\gamma^\alpha \frac{\not p_1 + \not q + m}{(p_1+q)^2 - m^2} \approx \varepsilon^*_\alpha(q)\,\bar{u}(p_1)\frac{2p_1^\alpha}{2p_1\cdot q} = \frac{p_1\cdot\varepsilon^*}{p_1\cdot q}\,\bar{u}(p_1)$$

The soft-photon limit of the corresponding term in $\mathcal{M}_1$ from radiation of γ from the μ^+ line is

$$\varepsilon^*_\alpha(q)\,\frac{-\not p_2 - \not q + m}{(p_2+q)^2 - m^2}\gamma^\alpha v(p_2) = -\frac{p_2.\varepsilon^*(q)}{p_2.q}v(p_2).$$

H. Express $\mathcal{M}_1$ in the soft-photon limit as a scalar factor multiplying $\mathcal{M}_0$.

$$\mathcal{M}_1 \simeq e\,\varepsilon_\mu(P)\,\varepsilon^*_\alpha(q)\,\bar{u}(p_1)\left(\frac{2p_1^\alpha}{2p_1\cdot q}(g_V - g_A\gamma_5)\gamma^\mu + (g_V - g_A\gamma_5)\gamma^\mu\frac{-2p_2^\alpha}{p_2\cdot q}\right)v(p_2)$$

$$= e\left(\frac{p_1\cdot\varepsilon(q)}{p_1\cdot q} - \frac{p_2\cdot\varepsilon(q)}{p_2\cdot q}\right)\mathcal{M}_0$$