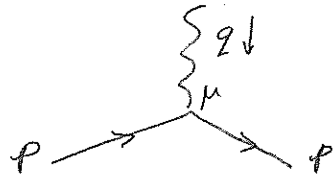
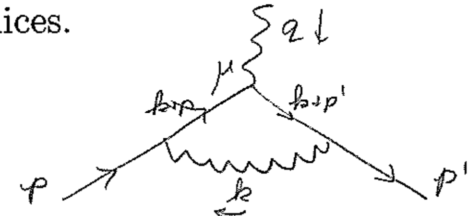
QED Vertex Function

The complete QED vertex $\Gamma^{\mu}_{ij}(p',p)$ is the 1PI Green function for an incoming electron of momentum p, an outgoing electron of momentum p', and an incoming photon of momentum $q = p' - p$.

A. The tree diagram for $ie\Gamma^{\mu}(p',p)$ is the QED vertex. Draw the diagram, labeling the momenta and the Lorentz and spinor indices.



B. Draw the one-loop diagram for $ie\Gamma^{\mu}(p',p)$, labelling the momenta and the Lorentz and spinor indices.



C. Identify 3 independent nontrivial scalars that can be constructed from the 4-vectors p and p'.

$p^2, \; p'^2, \; p \cdot p'$

Suppose the electrons are on shell: $p^2 = m_e^2$ and $p'^2 = m_e^2$.
D. Identify the nontrivial scalar that can be constructed from p and p' ~~that is odd under interchange of p and p'~~.

$p \cdot p'$ or $q^2 = 2m^2 - 2pp'$

The 16 matrices $\mathbb{1}$, γ^{μ}, $\sigma^{\mu\nu}$, $\gamma^{\mu}\gamma^5$, and γ^5 are a basis for 4×4 Dirac matrices.
E. Express $\Gamma^{\mu}(p',p)$ as a linear combination of these basis matrices with coefficients that are Lorentz-tensor functions of p and p'.

$$\Gamma^{\mu}(p',p) = A^{\mu}(p',p)\mathbb{1} + B^{\mu}{}_{\nu}(p',p)\gamma^{\mu} + C^{\mu}{}_{\nu\lambda}(p',p)\,\sigma^{\nu\lambda} + D^{\mu}{}_{\nu}(p',p)\,\gamma^{\nu}\gamma_5 + E^{\mu}(p',p)\,\gamma_5$$

Lorentz invariance implies that $A^{\mu}(p',p)$ can be expanded in terms of p^{μ} and p'^{μ} (or $(p+p')^{\mu}$ and q^{μ}) with coefficients that are scalar functions of p and p':

$$A^{\mu}(p',p) = A_1(p'^2, p^2, q^2)\,(p+p')^{\mu} + A_2(p'^2, p^2, q^2)\,q^{\mu}.$$

F. Expand $B^{\mu\nu}(p',p)$ in terms of Lorentz tensors constructed from p and p' with scalar coefficient functions.

$$B^{\mu\nu}(p',p) = B_0(p'^2, p^2, q^2)\,g^{\mu\nu} + B_1(p'^2, p^2, q^2)\,p^{\mu}p^{\nu} + B_2(p'^2, p^2, q^2)\,p^{\mu}p'^{\nu} + B_3(p^2, p'^2, q^2)\,p'^{\mu}p^{\nu} + B_4(p'^2, p^2, q^2)\,p'^{\mu}p'^{\nu}$$

An electron current can be obtained by putting the electron lines in $\Gamma^\mu(p',p)$ on shell ($p^2 = m_e^2$, $p'^2 = m_e^2$) and sandwiching it between electron spinors:

$$J^\mu(p',p) = \bar{u}(p')\,\Gamma^\mu(p',p)\,u(p).$$

The Dirac equations for the electron spinors are

$$(\not{p} - m)u(p) = 0, \qquad \bar{u}(p')(\not{p}' - m) = 0.$$

The term $A^\mu{}_\nu(p',p)\mathbb{1}$ in $\Gamma^\mu(p',p)$ is a linear combination of $(p+p')^\mu\mathbb{1}$ and $q^\mu\mathbb{1}$. The term $B^\mu{}_\nu(p',p)\gamma^\mu$ in $\Gamma^\mu(p',p)$ can be reduced to a linear combination of

$$\gamma^\mu, \quad (p+p')^{\overset{\mu}{\cancel{\nu}}}(\not{p}+\not{p}'), \quad (p+p')^\mu\not{q}, \quad q^\mu(\not{p}+\not{p}'), \quad q^\mu\not{q}.$$

G. Show that when these are sandwiched between the electron spinors, two of them are 0 and two of them reduce to the forms $(p+p')^\mu\mathbb{1}$ and $q^\mu\mathbb{1}$.

$\bar{u}(p')[(p+p')^\mu(\not{p}+\not{p}')]u(p) = 2m\,(p+p')^\mu\,\bar{u}(p')u(p) \qquad \bar{u}(p')[(p+p')^\mu\not{q}]u(p) = 0$

$\bar{u}(p')[q^\mu(\not{p}+\not{p}')]u(p) = 2m\,q^\mu\,\bar{u}(p')u(p) \qquad \bar{u}(p')[q^\mu\not{q}]u(p) = 0$

After exploiting parity symmetry, the most general form of the current is

$$J^\mu(p',p) = \bar{u}(p')\left[G_S(q^2)\,(p+p')^\mu + G_S'(q^2)\,q^\mu + G_V(q^2)\,\gamma^\mu + \frac{i}{2m_e}G_T(q^2)\,\sigma^{\mu\nu}q_\nu\right]u(p).$$

Gauge invariance implies the Ward identity

$$q_\mu\,\bar{u}(p')\,\Gamma^\mu(p',p)\,u(p) = 0.$$

H. Use the Ward identity to prove that $G_S'(q^2) = 0$.

$q_\mu\cdot(p+p')^\mu = (p'-p)\cdot(p'+p) = p'^2 - p^2 = 0 \qquad q_\mu\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\not{q}u(p) = 0$

$q_\mu\sigma^{\mu\nu}q_\nu = 0 \quad \Longrightarrow \quad q_\mu J^\mu(p',p) = q^2\,G_S(q^2)\,\bar{u}(p')u(p) = 0 \Longrightarrow G_S(q^2) = 0$

The Gordon decomposition of the matrix element of γ^μ between electron spinors is

$$\bar{u}(p')\,\gamma^\mu\,u(p) = \bar{u}(p')\left[(p+p')^\mu + \frac{i}{2m_e}\sigma^{\mu\nu}q_\nu\right]u(p).$$

I. Use the Gordon decomposition to reduce the electron current to the form

$$J^\mu(p',p) = \bar{u}(p')\left[F_1(q^2)\,\gamma^\mu + \frac{i}{2m_e}F_2(q^2)\,\sigma^{\mu\nu}q_\nu\right]u(p).$$

$$J^\mu(p',p) = \bar{u}(p')\left[G_S(q^2)\left(\gamma^\mu - \frac{i}{2m_e}\sigma^{\mu\nu}q_\nu\right) + G_V(q^2)\gamma^\mu + \frac{i}{2m_e}G_T(q^2)\sigma^{\mu\nu}q_\nu\right]u(p)$$

$$= \bar{u}(p')\left[\left(G_S(q^2) + G_V(q^2)\right)\gamma^\mu + \frac{i}{2m_e}\left(G_T(q^2) - G_S(q^2)\right)\frac{i}{2m_e}\sigma^{\mu\nu}q_\nu\right]u(p)$$