Ward Identity for Photon Propagator

The Lagrangian for QED with a covariant gauge-fixing term is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi - \tfrac{1}{2\xi}(\partial^\mu A_\mu)^2.$$

A gauge transformation has the form

$$\begin{aligned} \psi(x) &\longrightarrow e^{+i\varepsilon(x)}\psi(x), \\ \bar{\psi}(x) &\longrightarrow e^{-i\varepsilon(x)}\bar{\psi}(x), \\ A_\mu(x) &\longrightarrow A_\mu(x) - \tfrac{1}{e}\partial_\mu\varepsilon(x). \end{aligned}$$

A. The only term in $\mathcal{L}$ that is not invariant under gauge transformations is the gauge-fixing term. What is the change $\delta\mathcal{L}$ in $\mathcal{L}$ at first order in ε?

$$\delta\mathcal{L} = -\frac{1}{2\xi}\,2\,(\partial^\mu A_\mu)\,\partial^\nu\left(-\frac{1}{e}\partial_\nu\epsilon\right) = \frac{1}{\xi e}\,\partial^\mu A_\mu\,\partial^2\epsilon$$

B. The first-order change in the action from a gauge transformation is

$$\delta S = \int d^4x\,\left(\tfrac{1}{\xi e}\partial^\mu A_\mu\,\partial^2\varepsilon\right).$$

Using integration by parts, express δS as an integral over x with an explicit factor of $\varepsilon(x)$.

$$\delta S = \int d^4x\;\epsilon\left(\frac{1}{\xi e}\,\partial^2\partial^\mu A_\mu\right)$$

C. The first-order change in $A_\nu(y)$ is $\delta A_\nu(y) = -\frac{1}{e}\partial_\nu\varepsilon(y)$. Using a delta function, express $\delta A_\nu(y)$ as an integral over x with an explicit factor of $\varepsilon(x)$.

$$\delta A_\nu(y) = \int d^4x\left(-\frac{1}{e}\partial_\nu\epsilon(x)\right)\delta^4(x-y) = \int d^4x\;\epsilon(x)\left(\frac{1}{e}\partial_{x\nu}\delta^4(x-y)\right)$$

D. The first order change in $e^{iS}A_\nu(y)$ is

$$\delta\big(e^{iS}A_\nu(y)\big) = e^{iS}[i\delta S\,A_\nu(y) + \delta A_\nu(y)].$$

Express this as an integral over x with an explicit factor of $\varepsilon(x)$.

$$\delta\left(e^{iS}A_\nu(y)\right) = e^{iS}\int d^4x\;\epsilon(x)\left[\frac{i}{\xi e}\,\partial^\mu\partial^2 A_\mu(x)\,A_\nu(y) + \frac{1}{e}\partial_{x\nu}\delta^4(x-y)\right]$$

The first order change in $e^{iS}A_\nu(y)$ under a gauge transformation can be expressed as

$$\delta\big(e^{iS}A_\nu(y)\big) = e^{iS}\int \mathrm{d}^4x\,\varepsilon(x)\left[\frac{i}{\xi e}\partial^\mu\partial^2 A_\mu(x)A_\nu(y) + \frac{1}{e}\frac{\partial}{\partial x^\nu}\delta^4(x-y)\right].$$

The path integral weighted by a photon field is

$$\textstyle\int \mathrm{D}A \int \mathrm{D}\bar\psi \mathrm{D}\psi\, e^{iS}\, A_\nu(y).$$

The path integral is invariant under gauge transformations.

E. The measure of the path integral is the product over spacetime points x of

$$\prod_{\mu=0}^{3}\int_{-\infty}^{+\infty}\mathrm{d}A_\mu(x)\prod_{i=1}^{4}\int \mathrm{d}\bar\psi_i(x)\mathrm{d}\psi_i(x).$$

Show that this measure is invariant under a gauge transformation.

$\prod\int dA_\mu$ is invariant, because after the shift in A_μ, it is still an integral from $-\infty$ to $+\infty$

$\prod\int d\bar\Psi_i\, d\Psi_i$ is invariant, because the phases $e^{-i\varepsilon}$ and $e^{+i\varepsilon}$ in the Jacobian cancel.

F. Express the first-order change in the path integral from a gauge transformation as an integral over x with an explicit factor of $\varepsilon(x)$.

$$\textstyle\delta\big(\int \mathrm{D}A \int \mathrm{D}\bar\psi \mathrm{D}\psi\, e^{iS}A_\nu(y)\big) = \int \mathrm{D}A \int \mathrm{D}\bar\psi \mathrm{D}\psi\, \delta\big(e^{iS}A_\nu(y)\big)$$

$$= \int \mathrm{d}^4x\,\varepsilon(x)\times \int \mathcal{D}A \int \mathcal{D}\bar\Psi\mathcal{D}\Psi\, e^{iS}\left[\frac{i}{\xi e}\partial^\mu\partial^2 A_\mu(x)A_\nu(y) + \frac{1}{e}\frac{\partial}{\partial x^\nu}\delta^4(x-y)\right]$$

G. The first-order change in the path integral must be zero for all functions $\varepsilon(x)$. Deduce that a function of x involving path integrals is equal to zero.

$$\frac{i}{\xi e}\partial_x^\mu\partial_x^2 \int\mathcal{D}A\int\mathcal{D}\bar\Psi\mathcal{D}\Psi\, e^{iS}A_\mu(x)A_\nu(y) + \frac{1}{e}\frac{\partial}{\partial x^\nu}\delta^4(x-y)\int\mathcal{D}A\int\mathcal{D}\bar\Psi\mathcal{D}\Psi\, e^{iS} = 0$$

H. By dividing each term in the equation by the unweighted path integral, obtain the Ward identity for the photon propagator $\langle A_\mu(x)\,A_\nu(y)\rangle$.

$$\frac{i}{\xi e}\partial^\mu\partial^2\big\langle A_\mu(x)A_\nu(y)\big\rangle + \frac{1}{e}\frac{\partial}{\partial x^\nu}\delta^4(x-y) = 0$$