Regularization Method

for quantum field theory

loop integral : $\int d^4k$

ultraviolet cutoff Λ : limit $\Lambda \to \infty$?

desirable properties:

- allow shifts: $k^\mu \to k^\mu + q^\mu$
- respect symmetries: rotational invariance
  Lorentz invariance
  gauge invariance
  supersymmetry

1. energy and momentum cutoffs

$$\int d^4k \longrightarrow \int_{-\Lambda}^{+\Lambda} dk_0 \int_{-\Lambda}^{+\Lambda} dk_x \int_{-\Lambda}^{+\Lambda} dk_y \int_{-\Lambda}^{+\Lambda} dk_z$$

allows shifts only if at most log-divergent

violates rotational symmetry, ...

invariant mass cutoff?

$$\int d^4k \longrightarrow \int dk_0 \int d^3k \quad |k_0^2 - \vec{k}^2| < \Lambda^2$$

does not regularize integrals over $\vec{k}$

2. Euclidean momentum cutoff

$$\int d^4k \longrightarrow i\int dk_4 \int d^3k \quad |k_4^2 + \vec{k}^2| < \Lambda^2$$

allows shifts: $k^\mu \to k^\mu + q^\mu$

respects rotational, Lorentz invariance
violates gauge invariance

3. dimensional regularization

$$\int d^4k \longrightarrow \int d^{4-2\epsilon}k$$

allows shifts: $k^\mu \to k^\mu + q^\mu$

respects rotational, Lorentz, gauge invariance
violates supersymmetry

Euclidean Momentum Cutoff Regularization

Minkowski 4-momentum: $k^\mu = (k_0, \vec{k})$

$$k^2 = k_0^2 - \vec{k}^2$$

propagator: $\dfrac{i}{k^2 - m^2 + i\epsilon} = \dfrac{i}{k_0^2 - \vec{k}^2 - m^2 + i\epsilon}$

Euclidean 4-momentum: $k_E^\mu = (\vec{k}, k_4)$

$$k_E^2 = \vec{k}^2 + k_4^2$$

propagator: $\dfrac{1}{k_E^2 + m^2} = \dfrac{1}{\vec{k}^2 + k_4^2 + m^2}$

loop integral: $\displaystyle\int \frac{d^4k}{(2\pi)^4} F(k) = \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} F(k)$

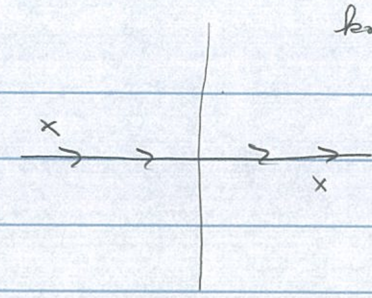
step 0: combine propagators into single denominator using Feynman parameter

shift loop momentum to eliminate terms in denominator linear in k^μ:

$k \to k'$

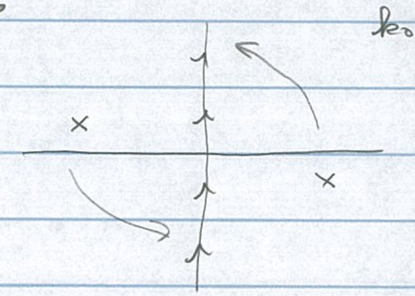
$$= \int \frac{d^4k}{(2\pi)^4} F(k')$$

energy integral: $\int_{-\infty}^{\infty} dk_0$



1. Wick rotate to imaginary axis

$$= \int_{-i\infty}^{+i\infty} dk_0$$



2. change variables to Euclidean energy: $k_4 = -ik_0$

$dk_0 = i\,dk_4$

$$= i\int_{-\infty}^{+\infty} dk_4$$

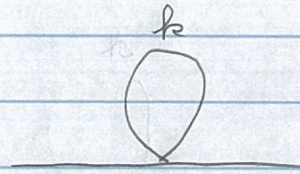
3. impose cutoff on Euclidean 4-momentum:

$$k_E^2 > \Lambda^2$$

4. Take limit $\Lambda \longrightarrow \infty$ where possible

- allows shift in loop momentum
- respects rotational, Lorentz symmetry
- violates gauge invariance, supersymmetry

Euclidean momentum cutoff regularization of simplest 1-loop integral



loop integral: $I = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}$

$$= \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{i}{k_0^2 - \vec{k}^2 - m^2 + i\epsilon}$$

1. Wick rotate k_0 integral to imaginary axis

$$\int_{-\infty}^{\infty} dk_0 = \int_{-i\infty}^{+i\infty} dk_0$$

2. Change variables to Euclidean energy: $k_E = -ik_0$

$$= i \int_{-\infty}^{+\infty} dk_E$$

$$I = i \int \frac{d^4k_E}{(2\pi)^4} \frac{i}{-k_E^2 - m^2}$$

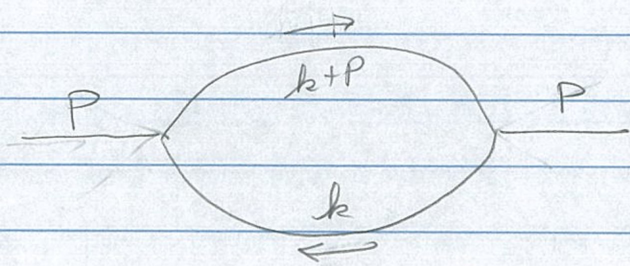
3. Impose cutoff on Euclidean 4-momenta: $k_E^2 < \Lambda^2$

$$= \frac{1}{(2\pi)^4} \int k_E^3 \, dk_E \int d\Omega_4 \frac{1}{k_E^2 + m^2}$$

4. Integrate over angles in 4 dimensions: $\int d\Omega_4 = 2\pi^2$

5. Integrate over k_E: $I = \frac{1}{16\pi^2}\left(\Lambda^2 - m^2 \log \frac{\Lambda^2 + m^2}{m^2}\right)$

Euclidean momentum cutoff regularization
of momentum-dependent loop integral



loop integral: $I(P^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2-m^2+i\epsilon][(k+P)^2-m^2+i\epsilon]}$

power-counting: $I \sim \int \frac{d^4k}{(k^2)^2} \sim \int \frac{k^3 dk}{k^4} \sim \int^{\Lambda} \frac{dk}{k} \sim \log \Lambda$

$\Longrightarrow$ logarithmically ultraviolet divergent

Lorentz scalar function of P^μ

$\Longrightarrow$ function of P^2

(provided regularization is Lorentz invariant
such as Euclidean momentum cutoff)

1. combine propagators using Feynman parameter

$$\frac{1}{ab} = \int_0^1 dx \, \frac{1}{[x a + (1-x) b]^2}$$

$$\frac{1}{[k^2 - m^2 + i\epsilon][(k+P)^2 - m^2 + i\epsilon]} = \int_0^1 dx \, \frac{1}{[k^2 + 2x P\cdot k + x P^2 - m^2 + i\epsilon]^2}$$

2. shift loop momentum so denominator is even function of k

$$k \longrightarrow k - xP$$

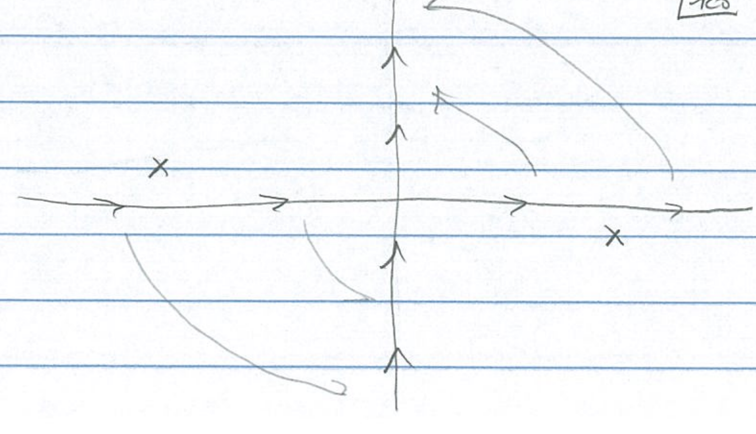
$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 + 2xP\cdot k + xP^2 - m^2 + i\epsilon]^2} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+xP)^2 - x^2P^2 + xP^2 - m^2 + i\epsilon]^2}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 + \underbrace{x(1-x)P^2 - m^2}_{-M^2(x)} + i\epsilon]^2}$$

3. Wick rotation:
deform k_0 contour from real axis to imaginary axis

$$\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{1}{[k_0^2 - \vec{k}^2 - M^2 + i\epsilon]^2} = \int_{-i\infty}^{+i\infty} \frac{dk_0}{2\pi} \frac{1}{[k_0^2 - \vec{k}^2 - M + i\epsilon]^2}$$

$\boxed{k_0}$



4. Change variable to Euclidean energy: $k_0 = ik_4$

$$\int_{-i\infty}^{+i\infty} \frac{dk_0}{2\pi} \frac{1}{[k_0^2 - \vec{k}^2 - M^2 + i\epsilon]^2} = i \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \frac{1}{[-k_4^2 - \vec{k}^2 - M^2]^2}$$

Integral over Euclidean space: $k_E = (k_1, k_2, k_3, k_4)$

$$k_E^2 = \vec{k}^2 + k_4^2$$

$$I(p^2) = i \int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k_E^2 \underbrace{- x(1-x)p^2 + m^2}_{M^2(x)}]^2}$$

5. Impose ultraviolet cutoff: $k_E^2 < \Lambda^2$

$$\int \frac{d^4k_E}{(2\pi)^4} \longrightarrow \frac{1}{(2\pi)^4} \int_0^\Lambda k_E^3 \, dk_E \int d\Omega_4$$

6. Integrate over angles in Euclidean space

$$\int d\Omega_4 = 2\pi^2$$

7. Integrate over k_E:

$$\int k_E^3 dk_E \frac{1}{[k_E^2 + M^2 - i\epsilon]^2} = \frac{1}{2}\left[\log \frac{\Lambda^2 + M^2}{M^2} - \frac{\Lambda^2}{\Lambda^2 + M^2}\right]$$

8. Take limit $\Lambda^2 >> |\mathcal{M}^2|$

$$\int k_E^3\, dk_E \frac{1}{[k_E^2+\mathcal{M}^2-i\epsilon]^2} = \frac{1}{2}\left[\log\frac{\Lambda^2}{m^2} - \log\frac{\mathcal{M}^2}{m^2} - 1\right]$$

$$I(p^2) = \frac{i}{16\pi^2}\int_0^1 dx \left[\log\frac{\Lambda^2}{m^2} - 1 - \log\frac{m^2 + x(1-x)(-p^2-i\epsilon)}{m^2}\right]$$

9. Integrate over Feynman parameter x

Special cases:

$p^2 = 0$: $\quad I(0) = \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{m^2} - 1\right]$

$p^2 = 4m^2$: $\quad I(4m^2) = \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{m^2} - 3\right]$

$p^2 = m^2$: $\quad I(m^2) = \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{m^2} - \frac{\pi}{\sqrt{3}} + 1\right]$

$|p^2| >> m^2$: $\quad I(p^2) \approx \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{m^2} - 1 - \left(\log\frac{-p^2-i\epsilon}{m^2} - 2\right)\right]$

$$= \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{-p^2} + 1\right] \qquad p^2<0$$

$$= \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{p^2} + 1 + i\pi\right] \qquad p^2>0$$