Grassmann Integrals

complex Grassmann variables θ, θ^+ satisfy

$$\theta^2 = \theta^{+2} = 0$$

$$\theta\theta^+ + \theta^+\theta = 0$$

Grassmann algebra generated by θ, θ^+ is 4-dimensional complex vector space consisting of element

$$a + b\theta + c\theta^+ + d\theta\theta^+, \quad a,b,c,d \in \mathbb{C}$$

"integration" over Grassmann variables

$$\int d\theta^+ d\theta \left(a + b\theta + c\theta^+ + d\theta\theta^+\right) = d$$

Gaussian integral

$$\int d\theta^+ d\theta \exp(-a\theta^+\theta)$$

$$= \int d\theta^+ d\theta \left(1 - a\theta^+\theta + \tfrac{1}{2}a^2\theta^+\theta\theta^+\theta + \cdots\right)$$

$$= \int d\theta^+ d\theta \left(1 - a\theta^+\theta\right)$$

$$= \int d\theta^+ d\theta \left(1 + a\theta\theta^+\right) = a$$

N complex Grassmann variables: $\theta_1, \theta_2, \ldots, \theta_N$

$\theta_1^+, \theta_2^+, \ldots, \theta_N^+$

all of which anti-commute

$$\theta_i \theta_j + \theta_j \theta_i = 0, \quad \theta_i \theta_j^+ + \theta_j^+ \theta_i = 0$$

Grassman algebra generated by $\{\theta_1, \ldots, \theta_N, \theta_1^+, \ldots, \theta_N^+\}$

2^{2N}-dimensional complex vector space with basis elements $1, \theta_i, \theta_i^+, \; i=1,\ldots,N$

$\theta_i \theta_j, \theta_i \theta_j^+, \theta_i^+ \theta_j^+, \; 1 \leq i < j \leq N$

$\vdots$

$\theta_1 \theta_1^+ \theta_2^+ \theta_2^+ \cdots \theta_N \theta_N^+$

integrals over all Grassmann variables

$$\int d\theta_1^+ d\theta_1 \cdots d\theta_N^+ d\theta_N \left(\theta_1 \theta_1^+ \theta_2 \theta_2^+ \cdots \theta_N \theta_N^+\right) = 1$$

integral is 0 for all other basis elements

Gaussian integral with hermitian matrix A:

$$\int d\theta_1^+ d\theta_1 \cdots d\theta_N^+ d\theta_N \exp\left(-\sum_{ij} \theta_i^+ A_{ij} \theta_j\right)$$

$$= \det A$$

N-index Levi-Civita tensor: $\epsilon_{n_1 \cdots n_N}$

$$\epsilon_{n_1 n_2 \cdots n_N} = +1 \text{ if } (n_1 n_2 \cdots n_N) \text{ is even permutation of } (12 \cdots N)$$

$$= -1 \quad \text{"} \quad \text{odd} \quad \text{"}$$

$$= 0 \quad \text{if any two indices are equal}$$

determinant of $N \times N$ matrix A

$$\det A = \frac{1}{N!} \epsilon_{i_1 i_2 \cdots i_N} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_N j_N} \epsilon_{j_1 j_2 \cdots j_N}$$

linear transformation of Grassmann coordinates with unitary matrix U

$$\theta_i \longrightarrow \sum_{i=1}^{N} U_{ij} \theta_j$$

integration measure

$$d\theta_1 d\theta_2 \cdots d\theta_N \longrightarrow (\det U)\, d\theta_1 d\theta_2 \cdots d\theta_N$$

verify:

$$d\theta_1 \cdots d\theta_N = \frac{1}{N!} \epsilon_{i_1 \cdots i_N} d\theta_{i_1} \cdots d\theta_{i_N}$$

$$\longrightarrow \frac{1}{N!} \epsilon_{i_1 \cdots i_N} (U_{i_1 j_1} d\theta_{j_1}) \cdots (U_{i_N j_N} d\theta_{j_N})$$

$$= \frac{1}{N!} \epsilon_{i_1 \cdots \epsilon_N} (U_{i_1 j_1} \cdots U_{i_N j_N}) \underbrace{d\theta_{j_1} \cdots d\theta_{j_N}}_{\epsilon_{j_1 \cdots j_N} d\theta_1 \cdots d\theta_N}$$

$$= \frac{1}{N!} \epsilon_{i_1 \cdots i_N} (U_{i_1 j_1} \cdots U_{i_N j_N}) \epsilon_{j_1 \cdots j_N} \, d\theta_1 \cdots d\theta_N$$

$$= (\det U) \, d\theta_1 \cdots d\theta_N$$

Gaussian integral

Hermitian matrix A can be diagonalized

by unitary transformation: $A = U \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{pmatrix} U^+$

$$\int d\theta_1^+ \cdots d\theta_N^+ d\theta_1 \cdots d\theta_N \exp\left(-\sum_{ij} \theta_i^+ A_{ij} \theta_j\right)$$

change variable: $\theta_i \rightarrow \sum_j U_{ij} \theta_j$, $\theta_i^+ \rightarrow \sum_j \theta_j^+ U_{ji}^+$

$$= (\det U^+)(\det U) \int d\theta_1^+ \cdots d\theta_N^+ d\theta_1 \cdots d\theta_N \exp\left(-\sum_{n=1}^{N} a_n \theta_n^+ \theta_n\right)$$

$$= a_1 \cdot a_2 \cdots a_N = \det A$$