

Functional Identities for scalar field

real scalar field theory

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \mathcal{V}_{\text{int}}(\phi)$$

classical equation of motion

$$(\square + m^2) \phi + \mathcal{V}'_{\text{int}}(\phi) = 0$$

Schwinger Dyson equation for Green function

$$\begin{aligned} & (\square_x + m^2) \langle T \hat{\phi}(x) \hat{\phi}(y_1) \cdots \hat{\phi}(y_n) \rangle \\ &= - \langle T \mathcal{V}'_{\text{int}}(\hat{\phi}(x)) \hat{\phi}(y_1) \cdots \hat{\phi}(y_n) \rangle \\ & \quad - i \delta^4(x - y_1) \langle T \phi(y_2) \cdots \phi(y_n) \rangle \\ & \quad \vdots \\ & \quad - i \delta^4(x - y_n) \langle T \phi(y_1) \cdots \phi(y_{n-1}) \rangle \end{aligned}$$

Derive from invariance of path integral
under shift $\phi(x) + \epsilon(x)$.

path integral

$$\int \mathcal{D}\phi \exp(iS[\phi]) \phi(y_1) \dots \phi(y_n)$$
$$= \int \mathcal{D}\phi \exp(i \int d^4x [-\frac{1}{2} \phi (\square + m^2) \phi - \mathcal{V}_{int}(\phi)]) \phi(y_1) \dots \phi(y_n)$$

invariant under $\phi(x) \rightarrow \phi(x) + \epsilon(x)$

$$= \int \mathcal{D}\phi \exp(i \int d^4x [-\frac{1}{2} (\phi + \epsilon) (\square + m^2) (\phi + \epsilon) - \mathcal{V}_{int}(\phi + \epsilon)])$$
$$\times [\phi(y_1) + \epsilon(y_1)] \dots [\phi(y_n) + \epsilon(y_n)]$$

expand both sides to 1st order in $\epsilon(x)$

$$0 = \int \mathcal{D}\phi \exp(iS[\phi]) \left(-i \int d^4x \epsilon(x) [\square + m^2] \phi(x) + \mathcal{V}'_{int}(\phi(x)) \right) \phi(y_1) \dots \phi(y_n$$
$$+ \epsilon(y_1) \phi(y_2) \dots \phi(y_n) + \dots + \phi(y_1) \dots \phi(y_{n-1}) \epsilon(y_n)$$

express right side with factor $\int d^4x \epsilon(x)$

$$0 = \int d^4x \epsilon(x) \int \mathcal{D}\phi \exp(iS[\phi])$$
$$\times \left(-i [\square_x + m^2] \phi(x) + \mathcal{V}'_{int}(\phi(x)) \right) \phi(y_1) \dots \phi(y_n$$
$$+ \delta(x - y_1) \phi(y_2) \dots \phi(y_n) + \dots + \delta(x - y_n) \phi(y_1) \dots \phi(y_{n-1})$$

If this is true for all infinitesimal functions $\epsilon(x)$,
 the function multiplying $\epsilon(x)$ must be 0

$$0 = \int \mathcal{D}\phi \exp(iS[\phi]) \left(-i[(\square_x + m^2)\phi(x) + \mathcal{V}_{int}'(\phi(x))] \phi(y_1) \dots \phi(y_n) \right. \\ \left. + \delta(x-y_1) \phi(y_2) \dots \phi(y_n) + \dots + \delta(x-y_n) \phi(x_1) \dots \phi(x_{n-1}) \right)$$

Divide by $\int \mathcal{D}\phi \exp(iS[\phi])$
 to get equation for correlator:

$$+i(\square_x + m^2) \langle T \hat{\phi}(x) \phi(y_1) \dots \phi(y_n) \rangle \\ = -i \langle T \mathcal{V}_{int}'(\hat{\phi}(x)) \phi(y_1) \dots \phi(y_n) \rangle \\ + \delta(x-y_1) \langle T \phi(y_2) \dots \phi(y_n) \rangle \\ \vdots \\ + \delta(x-y_n) \langle T \phi(y_1) \dots \phi(y_{n-1}) \rangle$$

diagrammatic representation for $\mathcal{V}_{int}(x) = \frac{1}{4!} g \phi^4$

$$i(\square_x + m^2) \times \text{diagram} = \frac{1}{3!} g \times \text{diagram} \\ + \delta^4(x-y_1) \text{diagram} + \dots + \delta^4(x-y_n) \text{diagram}$$

The diagrams consist of a shaded circle with external lines. The first diagram has an incoming line from the left and two outgoing lines labeled y_1 and y_2 . The second diagram has an incoming line from the left and n outgoing lines labeled y_1 through y_n . The third diagram has two outgoing lines labeled y_2 and y_n . The fourth diagram has two outgoing lines labeled y_1 and y_{n-1} .

complex scalar field theory

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \mathcal{V}_{int}(\phi^* \phi)$$

phase symmetry: $\phi(x) \rightarrow e^{i\alpha} \phi(x)$
 $\phi^*(x) \rightarrow e^{-i\alpha} \phi^*(x)$

Noether's Theorem for classical field theory

$$\Rightarrow \text{conserved current: } j^\mu = i(\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi)$$

$$\partial_\mu j^\mu = 0$$

Schwinger-Dyson equation for quantum field theory

$$\partial_\mu \langle T j^\mu(x) \phi^*(y_1) \phi(y_2) \rangle$$

$$= \delta(x-y_1) \langle T \phi^*(x) \phi(y_2) \rangle$$

$$+ \delta(x-y_2) \langle T \phi^*(y_1) \phi(x) \rangle$$

Derive from invariance of path integral
under spacetime dependent phase transformations

$$\phi(x) \rightarrow e^{i\epsilon(x)} \phi(x)$$

$$\phi^*(x) \rightarrow e^{-i\epsilon(x)} \phi^*(x)$$

path integral

$$\int \mathcal{D}\phi^* \mathcal{D}\phi \exp(iS[\phi]) \phi^*(x_1) \phi(x_2)$$

$$= \int \mathcal{D}\phi^* \mathcal{D}\phi \exp\left(i \int d^4x \left[d_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \mathcal{V}_{int}(\phi^* \phi) \right]\right) \phi^*(x_1) \phi(x_2)$$

invariant under $\phi(x) \rightarrow e^{i\epsilon(x)} \phi(x)$
 $\phi^*(x) \rightarrow e^{-i\epsilon(x)} \phi^*(x)$

1st order changes: $\delta \int d^4x \left[-\phi^* (\square + m^2) \phi \right]$

$$\delta \int d^4x \left[d_\mu \phi^* \partial^\mu \phi \right] = \int d^4x \left[d_\mu \phi^* (i \partial^\mu \epsilon \phi) + (-i d_\mu \epsilon \phi^*) \partial^\mu \phi \right]$$

$$\delta \int d^4x \left[-m^2 \phi^* \phi - \mathcal{V}_{int}(\phi^* \phi) \right] = 0$$

$$\delta \left(\phi^*(x_1) \phi(x_2) \right) = \left(-i \epsilon(x_1) \phi^*(x_1) \right) \phi(x_2) + \phi^*(x_1) \left(i \epsilon(x_2) \phi(x_2) \right)$$

sum of 1st order changes = 0

$$0 = \int \mathcal{D}\phi^* \mathcal{D}\phi \exp(iS[\phi]) \left(i \int d^4x d_\mu \epsilon \left[-i \phi^* \partial^\mu \phi + i \partial^\mu \phi^* \phi \right] \right. \\ \left. - i \epsilon(x_1) \phi^*(x_1) \phi(x_2) + i \epsilon(x_2) \phi^*(x_1) \phi(x_2) \right)$$

express with overall factor of $\int d^4x \epsilon(x)$

$$0 = \int d^4x \epsilon(x) \int \mathcal{D}\phi^* \mathcal{D}\phi \exp(iS[\phi]) \times \left(-\partial_\mu \left[\phi^*(x) \partial^\mu \phi(x) - \partial^\mu \phi^*(x) \phi(x) \right] \phi(y_1) \phi(y_2) - i\delta^4(x-y_1) \phi^*(x) \phi(y_2) + i\delta^4(x-y_2) \phi^*(x) \phi(y_2) \right)$$

if true for all function $\epsilon(x)$,
coefficient of $\epsilon(x)$ must be 0

$$\int \mathcal{D}\phi^* \mathcal{D}\phi \exp(iS[\phi]) \left[\partial_\mu j^\mu(x) \phi^*(y_1) \phi(y_2) - \delta^4(x-y_1) \phi^*(x) \phi(y_2) + \delta^4(x-y_2) \phi^*(y_1) \phi(x) \right]$$

Divide by $\int \mathcal{D}\phi^* \mathcal{D}\phi \exp(iS[\phi])$
to get equation for correlators

$$\partial_\mu \langle T j^\mu(x) \phi(y_1) \phi(y_2) \rangle = \delta^4(x-y_1) \langle T \phi^*(x) \phi(y_2) \rangle - \delta^4(x-y_2) \langle T \phi^*(y_1) \phi(x) \rangle$$

diagrammatic representation

$$\frac{\partial}{\partial X^\mu} \left(\text{diagram: a shaded circle with an incoming line from the left at vertex x and two outgoing lines to the right at vertices y_1 and y_2 } \right) = \delta^4(x-y_1) \text{diagram: a shaded circle with an incoming line from the left at vertex x and one outgoing line to the right at vertex y_2 } - \delta^4(x-y_2) \text{diagram: a shaded circle with an incoming line from the left at vertex x and one outgoing line to the right at vertex y_1 }$$

$$\text{diagram: a shaded circle with an incoming line from the left and an outgoing line to the right} = \text{vertex for current at } x : j^\mu(x) = i[\phi^*(x) \partial^\mu \phi(x) - \partial^\mu \phi^*(x) \phi(x)]$$