Functional Identities for scalar field

real scalar field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V_{int}(\phi)$$

classical equation of motion

$$(\Box + m^2)\phi + V_{int}'(\phi) = 0$$

Schwinger Dyson equation for Green function

$$(\Box_x + m^2)\langle T\,\hat{\phi}(x)\,\hat{\phi}(y_1)\cdots\hat{\phi}(y_n)\rangle$$

$$= -\langle T\,V_{int}'(\hat{\phi}(x))\,\hat{\phi}(y_1)\cdots\hat{\phi}(y_n)\rangle$$

$$-i\,\delta^4(x-y_1)\,\langle T\,\phi(y_2)\cdots\phi(y_n)\rangle$$

$$\vdots$$

$$-i\,\delta^4(x-y_n)\,\langle T\,\phi(y_1)\cdots\phi(y_{n-1})\rangle$$

Derive from invariance of path integral under shifts $\phi(x) + \epsilon(x)$.

path integral

$$\int \mathcal{D}\phi \, \exp(iS[\phi]) \, \phi(y_1) \cdots \phi(y_n)$$

$$= \int \mathcal{D}\phi \, \exp\left(i\int d^4x \left[-\tfrac{1}{2}\phi(\Box + m^2)\phi - \mathcal{V}_{int}(\phi)\right]\right) \phi(y_1) \cdots \phi(y_n)$$

invariant under $\phi(x) \to \phi(x) + \epsilon(x)$

$$= \int \mathcal{D}\phi \, \exp\left(i\int d^4x \left[-\tfrac{1}{2}(\phi+\epsilon)(\Box + m^2)(\phi+\epsilon) - \mathcal{V}_{int}(\phi+\epsilon)\right]\right)$$

$$\times \left[\phi(y_1) + \epsilon(y_1)\right] \cdots \left[\phi(y_n) + \epsilon(y_n)\right]$$

expand both sides to 1st order in $\epsilon(x)$

$$0 = \int \mathcal{D}\phi \, \exp(iS[\phi]) \Big(-i\int d^4x \, \epsilon(x) \left[(\Box + m^2)\phi(x) + \mathcal{V}_{int}'(\phi(x))\right] \phi(y_1) \cdots \phi(y_n)$$

$$+ \epsilon(y_1)\phi(y_2) \cdots \phi(y_n) + \cdots + \phi(y_1) \cdots \phi(y_{n-1})\epsilon(y_n) \Big)$$

express right side with factor $\int d^4x \, \epsilon(x)$

$$0 = \int d^4x \, \epsilon(x) \int \mathcal{D}\phi \, \exp(iS[\phi])$$

$$\times \Big(-i\left[(\Box_x + m^2)\phi(x) + \mathcal{V}_{int}'(\phi(x))\right] \phi(y_1) \cdots \phi(y_n)$$

$$+ \delta(x-y_1)\phi(y_2) \cdots \phi(y_n) + \cdots + \delta(x-y_n)\phi(y_1) \cdots \phi(y_{n-1}) \Big)$$

If this is true for all infinitesimal functions $\epsilon(x)$, the function multiplying $\epsilon(x)$ must be 0

$$0=\int \mathcal{D}\phi \, \exp\left(iS[\phi]\right)\Big(-i\left[(\Box_x+m^2)\phi(x)+V_{int}'(\phi(x))\right]\phi(y_1)\cdots\phi(y_n)$$

$$+\delta(x-y_1)\phi(y_2)\cdots\phi(y_n)+\cdots+\delta(x-y_n)\phi(x_1)\cdots\phi(x_{n-1})\Big)$$

Divide by $\int \mathcal{D}\phi \, \exp(iS[\phi])$
to get equation for correlator:

$$+i(\Box_x+m^2)\left\langle T\hat{\phi}(x)\,\phi(y_1)\cdots\phi(y_n)\right\rangle$$

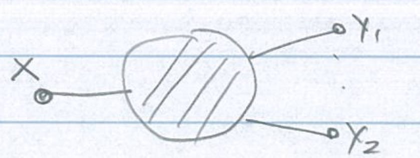
$$=-i\left\langle T\, V_{int}(\hat{\phi}(x))\,\phi(y_1)\cdots\phi(y_n)\right\rangle$$

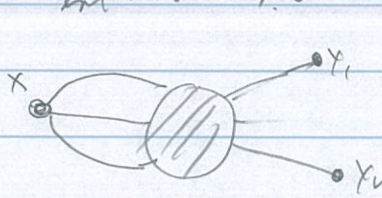
$$+\,\delta(x-y_1)\left\langle T\,\phi(y_2)\cdots\phi(y_n)\right\rangle$$

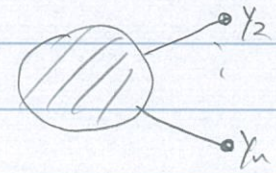
$$\vdots$$

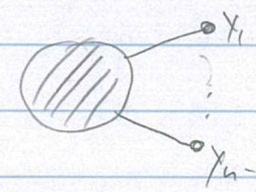
$$+\,\delta(x-x_n)\left\langle T\,\phi(y_1)\cdots\phi(y_n)\right\rangle$$

diagrammatic representation for $V_{int}(x)=\frac{1}{4!}g\phi^4$

$i(\Box_x+m^2)$ 

$=\frac{1}{3!}g$ 

$+\,\delta^4(x-y_1)$ 

$+\cdots+\,\delta^4(x-y_n)$ 

complex scalar field theory

$$\mathcal{L} = \partial_\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi - V_{int}(\phi^* \phi)$$

phase symmetry: $\phi(x) \longrightarrow e^{i\alpha} \phi(x)$

$$\phi^*(x) \to e^{-i\alpha} \phi^*(x)$$

Noether's Theorem for classical field theory

$\Longrightarrow$ conserved current: $j^\mu = i(\phi^* \partial_\mu \phi - \partial^\mu \phi^* \phi)$

$$\partial_\mu j^\mu = 0$$

Schwinger-Dyson equation for quantum field theory

$$\partial_\mu \langle T j^\mu(x) \phi^*(y_1) \phi(y_2) \rangle$$

$$= \delta(x-y_1) \langle T \phi^*(x) \phi(y_2) \rangle$$

$$- \delta(x-y_2) \langle T \phi^*(y_1) \phi(x) \rangle$$

Derive from invariance of path integral under <u>spacetime dependent</u> phase transformations

$$\phi(x) \longrightarrow e^{i\epsilon(x)} \phi(x)$$

$$\phi^*(x) \longrightarrow e^{-i\epsilon(x)} \phi^*(x)$$

path integral

$$\int \mathcal{D}\phi^* \mathcal{D}\phi \, \exp\left(i S[\phi]\right) \phi^*(y_1)\, \phi(y_2)$$

$$= \int \mathcal{D}\phi^* \mathcal{D}\phi \, \exp\left(i \int d^4x \left[\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - V_{int}(\phi^* \phi)\right]\right) \phi^*(y_1)\, \phi(y_2)$$

invariant under $\phi(x) \longrightarrow e^{i\epsilon(x)} \phi(x)$

$\phi^*(x) \longrightarrow e^{-i\epsilon(x)} \phi(x)$

1st order changes: $\delta \int d^4x \left[-\phi^* (\Box + m^2) \phi \right]$

$$\delta \int d^4x \left[\partial_\mu \phi^* \partial^\mu \phi \right] = \int d^4x \left[\partial_\mu \phi^* (i \partial^\mu \epsilon \phi) + (-i \partial_\mu \epsilon \phi^*) \partial^\mu \phi \right]$$

$$\delta \int d^4x \left[-m^2 \phi^* \phi - V_{int}(\phi^* \phi) \right] = 0$$

$$\delta \left(\phi^*(y_1)\, \phi(y_2) \right) = \left(-i \epsilon(y_1) \phi^*(y_1) \right) \phi(y_2) + \phi^*(y_1) \left(i \epsilon(y_2)\, \phi(y_2) \right)$$

sum of 1st order changes 0

$$0 = \int \mathcal{D}\phi^* \mathcal{D}\phi \, \exp\left(i S[\phi]\right) \Big(i \int d^4x \, \partial_\mu \epsilon \left[-i \phi^* \partial^\mu \phi + i \partial^\mu \phi^* \phi \right]$$

$$- i\, \epsilon(y_1) \phi^*(y_1) \phi(y_2) + i\, \epsilon(y_2) \phi^*(y_1) \phi(y_2) \Big)$$

express with overall factor of $\int d^4x \, \epsilon(x)$

$$0 = \int d^4x \, \epsilon(x) \int \mathcal{D}\phi^* \mathcal{D}\phi \; \exp(iS[\phi])$$

$$\times \Big(-\partial_\mu \left[i\,\phi^*(x)\,\partial^\mu \phi(x) - i\,\partial^\mu \phi^*(x)\,\phi(x) \right] \phi(y_1)\,\phi(y_2)$$

$$- i\delta^4(x-y_1)\,\phi^*(x)\,\phi(y_2) + i\delta^4(x-y_2)\,\phi^*(x)\,\phi(y_2) \Big)$$

if true for all function $\epsilon(x)$,
coefficient of $\epsilon(x)$ must be 0

$$\int \mathcal{D}\phi^* \mathcal{D}\phi \; \exp(iS[\phi]) \Big[\partial_\mu j^\mu(x)\,\phi^*(y_1)\,\phi(y_2)$$

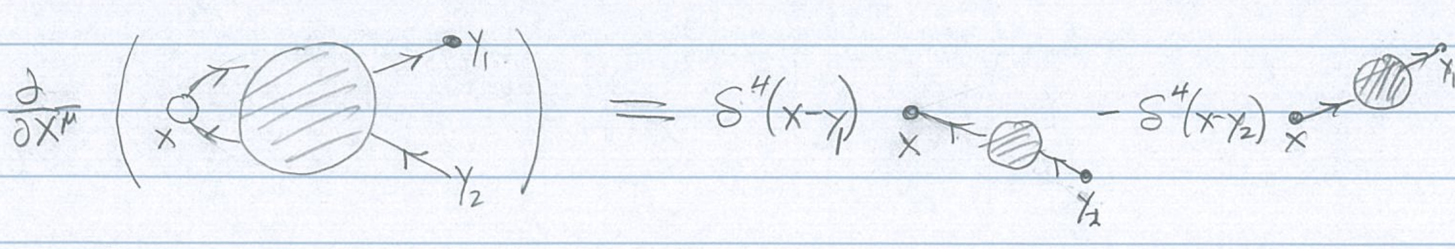
$$- \delta^4(x-y_1)\,\phi^*(x)\,\phi(y_2) + \delta^4(x-y_2)\,\phi^*(y_1)\,\phi(x) \Big]$$

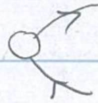
Divide by $\int \mathcal{D}\phi^* \mathcal{D}\phi \; \exp(iS[\phi])$
to get equation for correlators

$$\partial_\mu \langle T\, j^\mu(x)\,\phi(y_1)\,\phi(y_2) \rangle$$

$$= \delta^4(x-y_1) \langle T\, \phi^*(x)\,\phi(y_2) \rangle - \delta^4(x-y_2) \langle T\, \phi^*(y_1)\,\phi(x) \rangle$$

diagrammatic representation



 = vertex for current at x : $j^\mu(x) = i[\phi^*(x)\partial_\mu \phi(x) - \partial_\mu \phi^*(x)\,\phi(x)]$